1 Random Processes

There are many ways to define a random process, but for our purposes, the following is sufficient:

- A random process is a function of time $X(t)$, so that for each fixed time t^*, $X(t^*)$ is a random variable.

As a result, we can write the probability density function (pdf) of the random process at any given time. For example, $f_{X(t^*)}(x)$ represents the pdf of the random process at time t^*. Joint probability density functions measure the joint probability of the process at k different times; these are called *kth order statistics* of the random process. For example, for $k = 2$ and times t_1 and t_2, we can write the second order statistics as $f_{X(t_1),X(t_2)}(x_1, x_2)$.

1.1 Discrete time random processes

1.1.1 Definition, Mean, and Variance

It's easy to imagine a random process in discrete time, as merely a sequence of random variables, one for each time interval. For instance, consider the following two random processes defined at integer times $t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$:

Example 1 *At each time $t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, a fair coin is flipped. If the coin shows heads after the flip at time t, then $X(t) = 1$; otherwise, $X(t) = -1$. Thus, for any integer t^*, we can write*

$$f_{X(t^*)}(x) = \begin{cases} 0.5, & x = +1; \\ 0.5, & x = -1; \\ 0 & \text{otherwise.} \end{cases}$$

Since, at each fixed time t, the random process is a random variable, we can calculate the mean and variance of the process at each fixed time as usual for random variables. Thus, for the process as a whole, the mean and variance for a random process are calculated as functions of time. For instance, for the process in Example 1, the mean of this process is given by

$$\begin{aligned}\mu(t) &= \sum_{x\in\{+1,-1\}} x f_{X(t)}(x) \\ &= (+1)(0.5)+(-1)(0.5) \\ &= 0\end{aligned}$$

for all t. The variance of the process is given by

$$\begin{aligned}\sigma^2(t) &= \sum_{x\in\{+1,-1\}} (x-\mu(t))^2 f_{X(t)}(x) \\ &= (+1-0)^2(0.5)+(-1-0)^2(0.5) \\ &= 1\end{aligned}$$

for all t.

As an alternative, the following more compicated example has mean and variance that are non-trivial functions of time:

Example 2 *Let $X(0) = 0$. For each $t \in \{1, 2, \ldots\}$, a fair coin is flipped. If the coin shows heads after the flip at time t, then $X(t) = X(t-1) + 1$; otherwise, $X(t) = X(t-1)$.*

For any t, it is clear that $X(t)$ is the number of heads in the previous t trials. Such random variables are represented by the binomial distribution [1]. Thus,

$$f_{X(t)}(x) = \binom{t}{x}\frac{1}{2^t}.$$

The mean of this random process is given by

$$\mu(t) = \frac{t}{2},$$

and the variance is given by

$$\sigma^2(t) = \frac{t}{4}.$$

The reader is asked to prove these values in the exercises.

Instances of the random processes from Examples 1 and 2 are given in Figure 1.

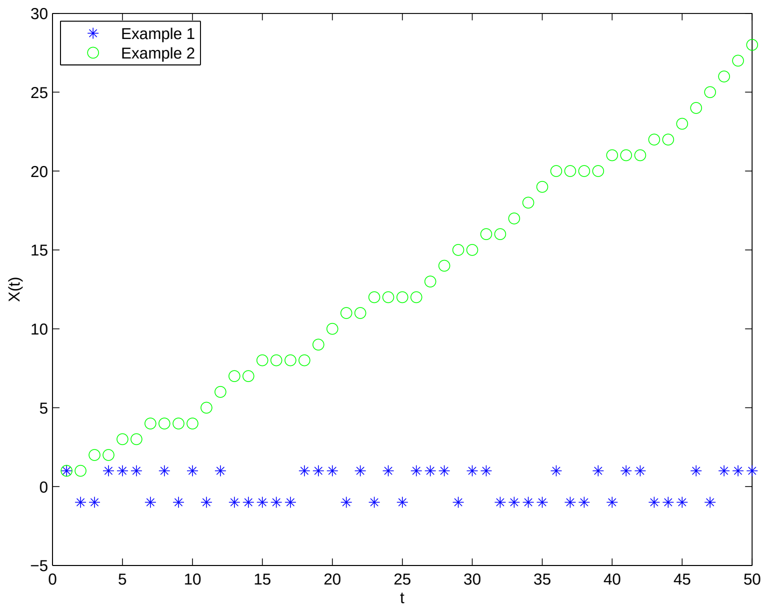

Figure 1: Illustration of the discrete-time random processes from Examples 1 and 2.

1.1.2 Autocorrelation

Suppose you wanted a measure of correlation between two random variables, X_1 and X_2, with the same mean $\mu = 0$ and the same variance $\sigma^2 > 0$. As a candidate for this measure, consider

$$R = E[X_1 X_2]. \tag{1}$$

If the random variables are independent (i.e., uncorrelated), then since $E[X_1 X_2] = E[X_1]E[X_2]$ for independent random variables, we would have

$$R = E[X_1]E[X_2] = \mu^2 = 0,$$

bearing in mind that each of the random variables are zero mean. On the other hand, if the two random variables are completely correlated (i.e., $X_1 = X_2$), we would have

$$R = E[X_1 X_2] = E[X_1^2] = \sigma^2.$$

Further, if they were completely anticorrelated (i.e., $X_1 = -X_2$), it is easy to see that $R = -\sigma^2$.

This measure of correlation also has the following nice property:

Theorem 1 *Given the above definitions,* $|R| \leq \sigma^2$*.*

Proof: Start with $E[(X_1 + X_2)^2]$. We can write:

$$\begin{aligned} E[(X_1 + X_2)^2] &= E[X_1^2 + 2X_1X_2 + X_2^2] \\ &= E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\ &= \sigma^2 + 2R + \sigma^2 \\ &= 2\sigma^2 + 2R. \end{aligned}$$

Since $(X_1+X_2)^2 \geq 0$ for all X_1 and X_2, it is true that $E[(X_1+X_2)^2] \geq 0$. Thus, $2\sigma^2+2R \geq 0$, so $R \geq -\sigma^2$. Repeating the same procedure but starting with $E[(X_1 - X_2)^2]$, we have that $R \leq \sigma^2$, and the theorem follows. ■

Since $R = 0$ when X_1 and X_2 are independent, $R = \sigma^2$ (the maximum possible value) when they are completely correlated, and $R = -\sigma^2$ (the minimum possible value) when they are completely anticorrelated, R is a good candidate for a correlation measure. The magnitude of R indicates the degree of correlation between X_1 and X_2, while the sign indicates whether the variables are correlated or anticorrelated. Properties of this correlation measure when the variances are unequal, or when the means are nonzero, are considered in the exercises.

We apply this correlation measure to different time instants of the same random process, which we refer to as the *autocorrelation*. In particular, let $X(t)$ be a discrete-time random process defined on $t \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Then the autocorrelation between $X(t_1)$ and $X(t_2)$ is defined as

$$R(t_1, t_2) = E[X(t_1)X(t_2)]. \tag{2}$$

Note the similarity with (1), since $X(t)$ is merely a random variable for each time t. For the same reason, $R(t_1, t_2)$ has all the same properties as R.

1.1.3 Stationary random processes

A *stationary* discrete-time random process is a process for which the statistics do not change with time. Formally, a process is stationary if and only if

$$f_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1,x_2,\ldots,x_k) = f_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_k+\tau)}(x_1,x_2,\ldots,x_k) \qquad (3)$$

for all $k \in \{1,2,\ldots\}$ and all $\tau \in \{\ldots,-2,-1,0,1,2,\ldots\}$. This does *not* imply that the process $X(t)$ is constant with respect to time, only that the statistical variation of the process is the same, regardless of when you examine the process. The process in Example 1 is stationary; intuitively, this is because we keep flipping the same unchanging coin, and recording the outcome in the same way at all t.

We now examine the effects of stationarity on the mean, variance, and autocorrelation of a discrete-time random process $X(t)$. The mean $\mu(t)$ is calculated as follows:

$$\begin{aligned} \mu(t) &= \int_x x f_{X(t)}(x)\mathrm{d}x \\ &= \int_x x f_{X(t+\tau)}(x)\mathrm{d}x \\ &= \mu(t+\tau), \end{aligned}$$

where the second line follows from the fact that $f_{X(t)} = f_{X(t+\tau)}$ for all $\tau \in \{\ldots,-2,-1,0,1,2,\ldots\}$. Thus, $\mu(t) = \mu(t+\tau)$ for all τ, so $\mu(t)$ must be a constant with respect to t. Using a similar line of reasoning, we can show that $\sigma^2(t)$ is a constant with respect to t. Thus, for stationary random processes, we will write $\mu(t) = \mu$ and $\sigma^2(t) = \sigma^2$ for all t.

For the autocorrelation, we can write

$$\begin{aligned} R(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{x_1}\int_{x_2} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1 \qquad (4) \\ &= \int_{x_1}\int_{x_2} x_1 x_2 f_{X(t_1+\tau),X(t_2+\tau)}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1. \qquad (5) \end{aligned}$$

Let $\tau = \tau' - t_1$. Substituting back into (5), we have

$$\begin{aligned} R(t_1,t_2) &= \int_{x_1}\int_{x_2} x_1 x_2 f_{X(t_1+\tau'-t_1),X(t_2+\tau'-t_1)}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1 \\ &= \int_{x_1}\int_{x_2} x_1 x_2 f_{X(\tau'),X(t_2-t_1+\tau')}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1. \qquad (6) \end{aligned}$$

However, in (6), since $X(t)$ is stationary, $f_{X(\tau'),X(t_2-t_1+\tau')}(x_1,x_2)$ does not change for any value of τ'. Thus, setting $\tau'=0$, we can write

$$R(t_1,t_2)=\int_{x_1}\int_{x_2} x_1x_2 f_{X(0),X(t_2-t_1)}(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1,$$

which is not dependent on the exact values of t_1 or t_2, but only on the difference t_2-t_1. As a result, we can redefine the autocorrelation function for stationary random processes as $R(t_2-t_1)$; further, reusing τ to represent this difference, we will usually write $R(\tau)$, where

$$R(\tau)=E[X(t)X(t+\tau)]$$

for all t.

The properties that $\mu(t)=\mu$, $\sigma^2(t)=\sigma^2$, and $R(t_1,t_2)=R(t_2-t_1)$ apply only to the first and second order statistics of the process $X(t)$. In order to verify whether a process is stationary, it is necessary to prove the condition (3) for every order of statistics. In general this is a difficult task. However, in some circumstances, only first and second order statistics are required. In this case, we define a *wide-sense stationary* (WSS) process as any process which satisfies the first and second order conditions of $\mu(t)=\mu$, $\sigma^2(t)=\sigma^2$, and $R(t_1,t_2)=R(t_2-t_1)$. We have shown that all stationary processes are WSS, but it should seem clear that a WSS process is not necessarily stationary.

1.2 Continuous time random processes

A continuous time random process $X(t)$ is defined over all $t\in\mathbb{R}$, where $\mathbb{R}$ represents the set of real numbers. Following the definition of a random variable that we gave at the beginning of this document, for each time instant $t^*\in\mathbb{R}$ in continuous time, $X(t^*)$ is a random variable.

It is straightforward to move from discrete-time random processes to continuous-time random processes. One approach – but by no means the only one – would be to linearly interpolate between the values of the discrete-time process. In the following example, we apply this approach to Example 1:

Example 3 *Let $X(t)$ be a continuous-time random process with the following properties. For times $t\in\{\ldots,-2,-1,0,1,2,\ldots\}$ (i.e., integer times), the process is defined as in Example*

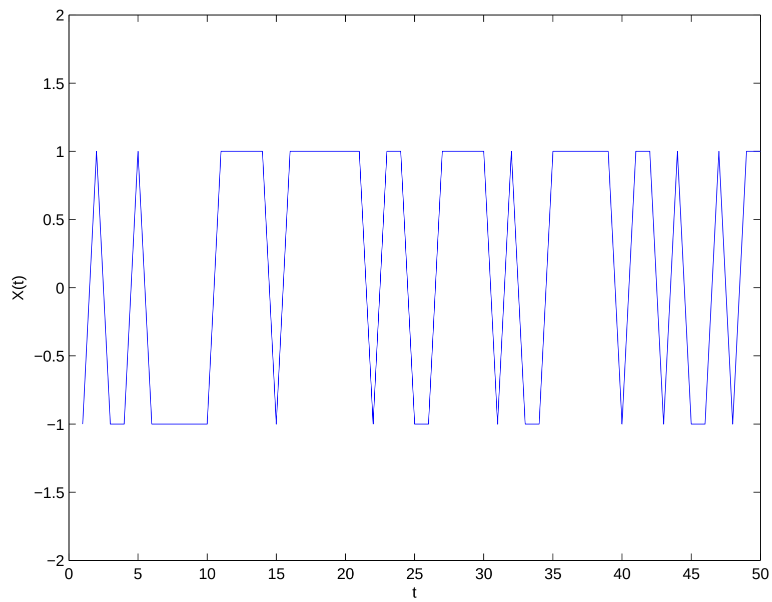

Figure 2: Illustration of the continuous-time random process from Example 3.

1. For all other t (i.e., non-integer times), letting $\lfloor t \rfloor$ represent the largest integer less than or equal to t, and letting $\lambda = t - \lfloor t \rfloor$,

$$X(t) = \lambda X(\lfloor t \rfloor) + (1 - \lambda) X(\lfloor t \rfloor + 1), \tag{7}$$

which linearly interpolates between the integer times. An illustration is given in Figure 2.

For integer times, $f_{X(t)}(x)$ is as given in Example 1. For non-integer times, from (7), $X(t)$ is a random variable which is dependent on $X(\lfloor t \rfloor)$ and $X(\lfloor t \rfloor + 1)$. Further, there are only four possible values of $X(t)$, corresponding to the four possible values of $X(\lfloor t \rfloor)$ and $X(\lfloor t \rfloor + 1)$ (i.e., the four possible values of those two coin flips):

$$f_{X(t)}(x) = \begin{cases} 1/4, & x = 1; \\ 1/4, & x = -1; \\ 1/4, & x = 2\lambda - 1; \\ 1/4, & x = 1 - 2\lambda; \\ 0 & \text{otherwise.} \end{cases}$$

In the special case where $\lambda = 0.5$, then $2\lambda - 1 = 1 - 2\lambda = 0$, and we have

$$f_{X(t)}(x) = \begin{cases} 1/4, & x = 1; \\ 1/4, & x = -1; \\ 1/2, & x = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Calculation of mean, variance, and autocorrelation is accomplished in exactly the same way as for discrete-time random processes, with the exception that time t can take any value in $\mathbb{R}$, not just integer values. Furthermore, stationary and wide-sense stationary processes are defined in the same way, again allowing $t \in \mathbb{R}$.

2 Exercises

1. For the random process in Example 2, show that $\mu(t) = t/2$, and $\sigma^2(t) = t/4$. Is this process stationary? Explain.

2. Suppose X_1 and X_2 are zero-mean random variables with variances σ_1^2 and σ_2^2, respectively. For the correlation measure R defined in (1), show that

$$|R| \leq \frac{\sigma_1^2 + \sigma_2^2}{2}.$$

3. Suppose X_1 and X_2 have the same *nonzero* mean μ, and the same variance σ^2. For the correlation measure R defined in (1), show that $|R| \leq \sigma^2 + \mu^2$.

4. Give an example of a discrete-time random process for which $\mu(t) = \mu$ and $\sigma^2(t) = \sigma^2$ for all t, but there exist t_1 and t_2 such that $R(t_1, t_2) \neq R(t_2 - t_1)$.

5. Calculate $\mu(t)$ and $R(t_1, t_2)$ for the continuous time random process given in Example 3. Is this process stationary? Explain.

6. Let $X(t) = X \sin(2\pi t)$, where X is a random variable corresponding to the result of a single fair coin flip: $X = 1$ if the coin is heads, and $X = -1$ is the coin is tails.

Does $X(t)$ satisfy the definition of a continuous-time random process? If so, calculate $f_{X(t)}(x)$; if not, explain why not.

References

[1] A. Leon-Garcia, *Probability and Random Processes for Electrical Engineering, 2nd ed.*, Reading, MA: Addison-Wesley, 1994.