ENG2200
Electric Circuits

Chapter 8

RLC Circuit

Natural and Step Response

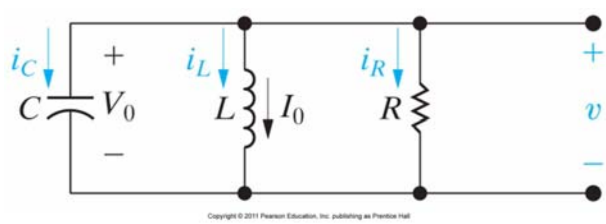
ENG2200 Topics to be covered

- Be able to determine the natural response of RLC circuits
- Be able to determine the step response of RLC circuits.

RLC Circuits

- The first step is to write either KVL or KCL for the circuit.
- Take the derivative to remove any integration
- Solve the resulting differential equation

Parallel RLC circuit



KCL
Sum of currents =0

$$i_R + i_L + i_c = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$$

Differentiating with respect to t, we get

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v}{L} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Solve this equation

Parallel RLC Circuits

- How to solve this differential equation?
- We can not separate the variables like we did with the RC or RL circuits.
- Without going into a lot of Math, we claim the solution will be in the form $v=Ae^{st}$
  - Exponential is the only function where high order of the derivatives have the same form (exponential)
  - First order (RL or RC) have the same form

Parallel RLC Circuits

- Assuming $v = Ae^{st}$ and substituting in the equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

$$v = Ae^{st}$$

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

Either A=0 (trivial solution) or
The quadratic part is 0

Parallel RLC Circuits

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Both of these 2 values satisfy the equation, their sum does.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

With a special case when $s_1 = s_2$

Parallel RLC Circuits

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

Neper frequency (α)

Resonant radian frequency (ω_0)

Damping ratio (ξ)

Parallel RLC Circuits

- The solution of the differential equation depends on the values of s1 and s2
- For simplicity assume A_1 and A_2 to be 1

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Consider three cases

Overdamped $\alpha > \omega_o$, $\xi > 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad , \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s = -\alpha \pm \omega_d \qquad \omega_d = \sqrt{\alpha^2 - \omega_0^2}$$

$$v(t) = A_1 e^{(-\alpha + \omega_d)t} + A_1 e^{(-\alpha - \omega_d)t}$$

Underdamped $\alpha < \omega_o$, $\xi < 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm j\omega_0\sqrt{1-\xi^2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s = -\alpha \pm j\omega_d \qquad \omega_d = \sqrt{\omega_0^2 - \alpha}$$

$$v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_1 e^{-\alpha t}\sin(\omega_d t)$$

Critically Damped $\alpha = \omega_o$, $\xi = 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

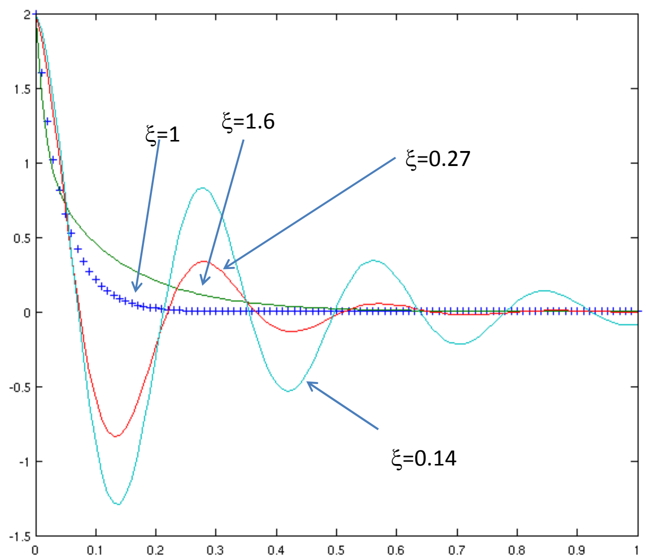
$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s_1 = s_1 = -\alpha$$

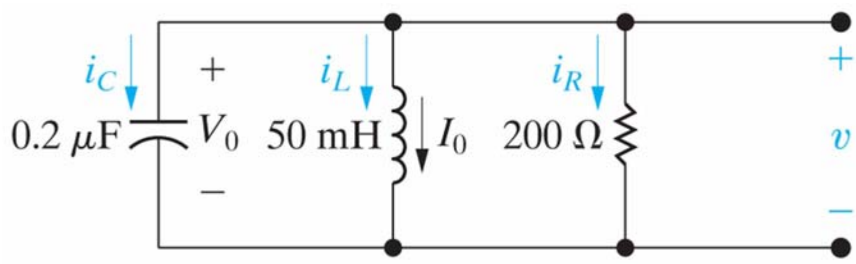
$$v(t) = D_1 t e^{-\alpha t} + D_1 e^{-\alpha t}$$



$A_1=A_2=B_1=B_2=D_1=D_2=1$

Parallel RLC Circuits

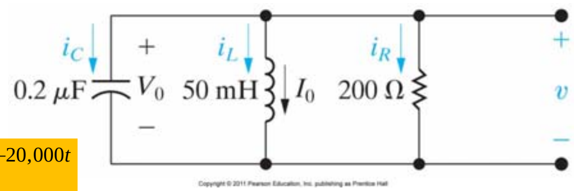
- Now we have a solution for the problem with



instantaneously

$$v_C(0^-) = v_C(0) = v_C(0^+)$$
$$i_L(0^-) = i_L(0) = i_L(0^+)$$

Example



$$v(t) = A_1 e^{-5{,}000t} + A_2 e^{-20{,}000t}$$

$$v(0^+) = 12 = A_1 + A_2$$

$v(0^+)$=12 V
$i_L(0^+)$=30 mA

Since the initial voltage across C (and R) is 12, the initial current in R =12/200=60 mA
$i_R(0^+)$= 60 mA

$$\frac{dv(0^+)}{dt} = -450{,}000V/s = -5000A_1 - 20{,}000A_2$$

$$i_C(0^+) = -i_R(0^+) - i_L(0^+) = -90mA$$

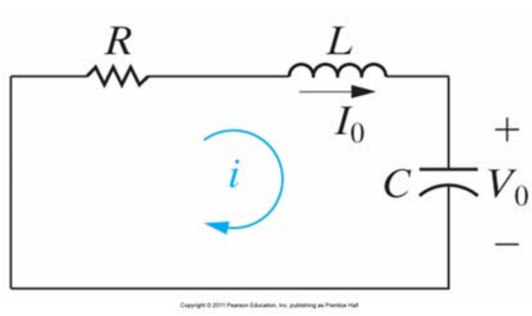
$$i_C(0^+) = C\frac{dv(0^+)}{dt}$$

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{-90mA}{0.2\mu F} = -450{,}000V/s$$

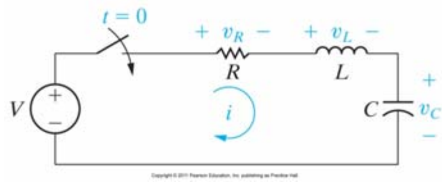
2 equation in 2 unknowns, solve to get A1 and A2

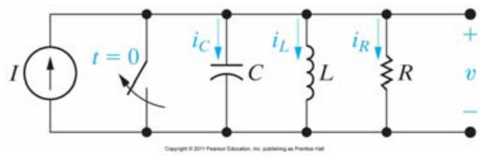
Series RLC

- Solved in the lab manual
- Only difference is α=R/2L



Step Response of RLC Circuit





$$v_L + v_C + v_R = V$$

$$L\frac{di}{dt} + v_c + iR = V$$

$$i = C\frac{dv_C}{dt}$$

$$LC\frac{d^2v_C}{dt^2} + v_C + RC\frac{dv_C}{dt} = V$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

$$i_L + i_C + i_R = I$$

$$i_L + C\frac{dv}{dt} + \frac{v}{R} = I$$

$$v = L\frac{di_L}{dt}$$

$$LC\frac{d^2i_L}{dt} + \frac{L}{R}\frac{di_L}{dt} + i_L = I$$

$$\frac{d^2i_L}{dt} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

Step Response of RLC Circuits

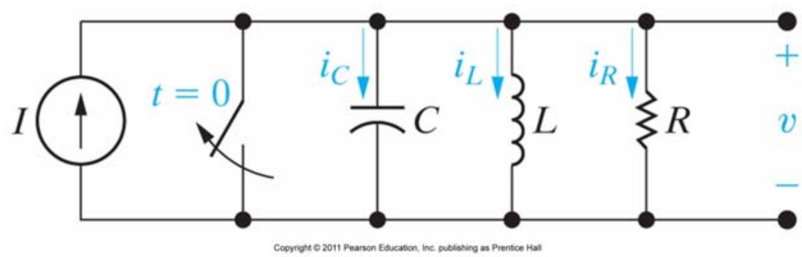
- A topic for a course in Math.
- Generally speaking, the solution of a second-order DE with a constant driving force equals the forced response plus the a response function identical to the natural response.

$$i = I_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as natural response} \end{Bmatrix}$$

$$v = V_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as natural response} \end{Bmatrix}$$

Example

- What is the final I_f



Example

- What is the final V_f