

ENG2200 Electric Circuits

Chapter 8
RLC Circuit
Natural and Step Response

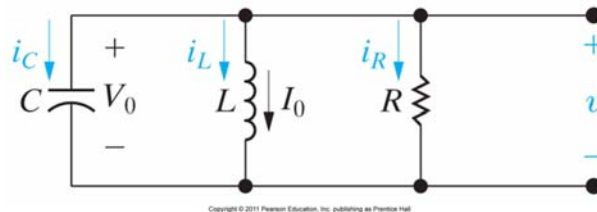
ENG2200 Topics to be covered

- Be able to determine the natural response of RLC circuits
- Be able to determine the step response of RLC circuits.

RLC Circuits

- The first step is to write either KVL or KCL for the circuit.
- Take the derivative to remove any integration
- Solve the resulting differential equation

Parallel RLC circuit



KCL
Sum of currents = 0

$$i_R + i_L + i_C = 0$$

$$\frac{v}{R} + \frac{1}{L} \int_0^t v d\tau + I_0 + C \frac{dv}{dt} = 0$$

Differentiating with respect to t, we get

$$C \frac{d^2v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{v}{L} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0$$

Solve this equation

Parallel RLC Circuits

- How to solve this differential equation?
- We can not separate the variables like we did with the RC or RL circuits.
- Without going into a lot of Math, we claim the solution will be in the form $v=Ae^{st}$
 - Exponential is the only function where high order of the derivatives have the same form (exponential)
 - First order (RL or RC) have the same form

Parallel RLC Circuits

- Assuming $v = Ae^{st}$ and substituting in the equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{v}{LC} = 0$$

$$v = Ae^{st}$$

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st} \left(s^2 + \frac{s}{RC} + \frac{1}{LC} \right) = 0$$

Either A=0 (trivial solution) or
The quadratic part is 0

Characteristic equation

Parallel RLC Circuits

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Both of these 2 values satisfy the equation, their sum does.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

With a special case when $s_1 = s_2$

The roots are determined by R, L, and C.
The initial conditions determine A_1, A_2

Parallel RLC Circuits

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi \omega_0 \pm \omega_0 \sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R} \sqrt{\frac{L}{C}}$$

Neper frequency

Resonant radian frequency

Damping ratio

Parallel RLC Circuits

- The solution of the differential equation depends on the values of s_1 and s_2
- For simplicity assume A_1 and A_2 to be 1

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- Consider three cases

Overdamping $\alpha > \omega_0$, $\zeta > 1$ -- 2 real roots

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

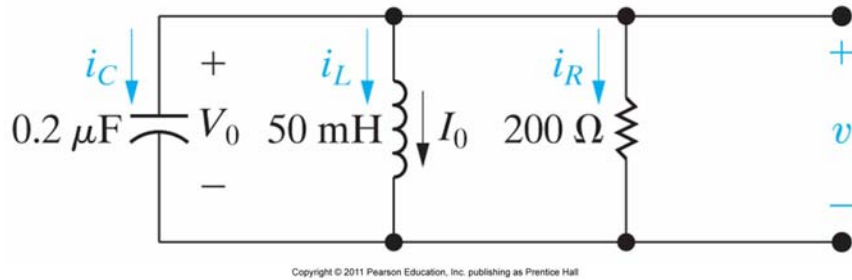
$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$v(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t}$$

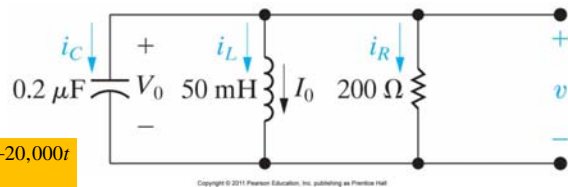
Parallel RLC Circuits



$$v_C(0^-) = v_C(0) = v_C(0^+)$$

$$i_L(0^-) = i_L(0) = i_L(0^+)$$

Example



$$v(t) = A_1 e^{-5,000t} + A_2 e^{-20,000t}$$

$$v(0^+) = 12 = A_1 + A_2$$

$$v(0^+) = 12 \text{ V}$$

$$i_L(0^+) = 30 \text{ mA}$$

$$i_C(0^+) = -i_R(0^+) - i_L(0^+)$$

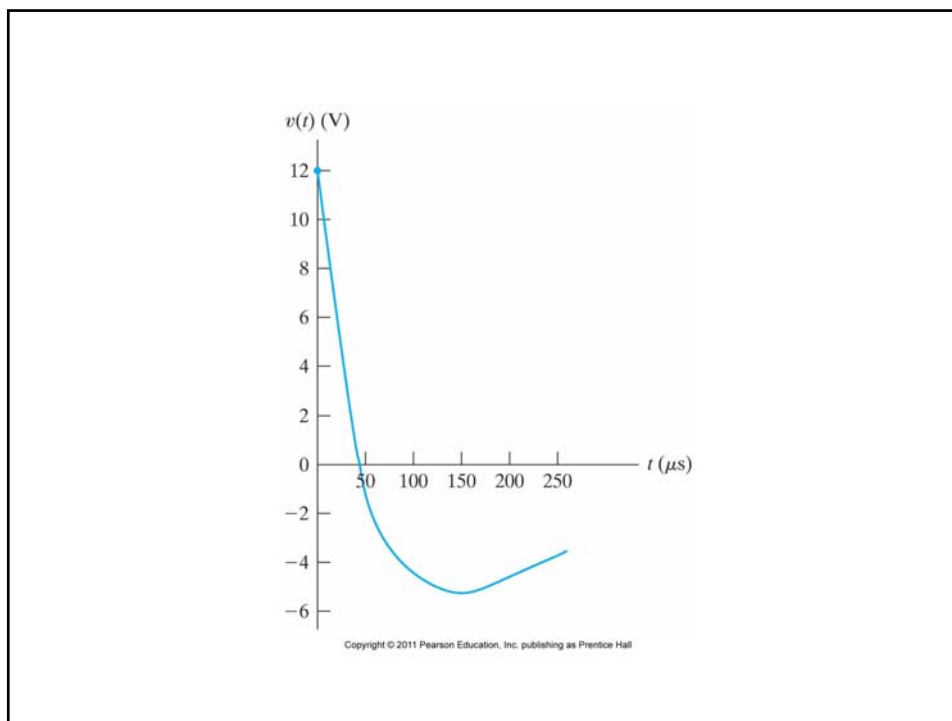
$$i_C(0^+) = C \frac{dv(0^+)}{dt}$$

$$C \frac{dv(0^+)}{dt} = -\frac{V_0}{R} - i_L(0^+)$$

$$\frac{dv(0^+)}{dt} = -\frac{V_0}{CR} - \frac{i_L(0^+)}{C}$$

$$\frac{dv(0^+)}{dt} = -450,000 \text{ V/s} = -5000A_1 - 20,000A_2$$

2 equation in 2 unknowns, solve to get A1 and A2



Underdamping $\alpha < \omega_0$, $\zeta < 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

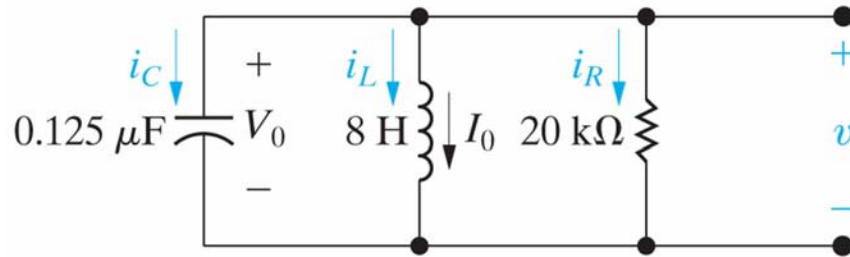
$$s_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

$$s = -\alpha \pm j\omega_d \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

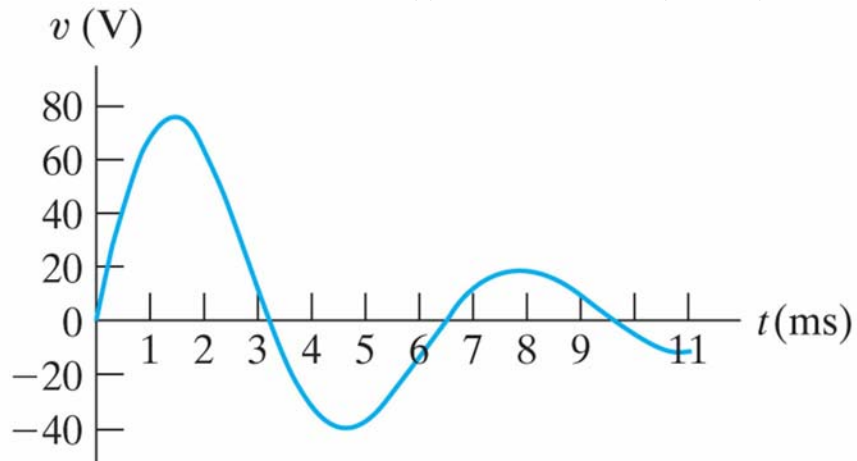
$$v(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$$

Example



$V_0=0$
 $I_0=-12.25\text{mA}$

$$v(t) = 100e^{-200t} \sin(979.8t)$$



Critically Damped $\alpha = \omega_0, \zeta = 1$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \text{ rad/s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

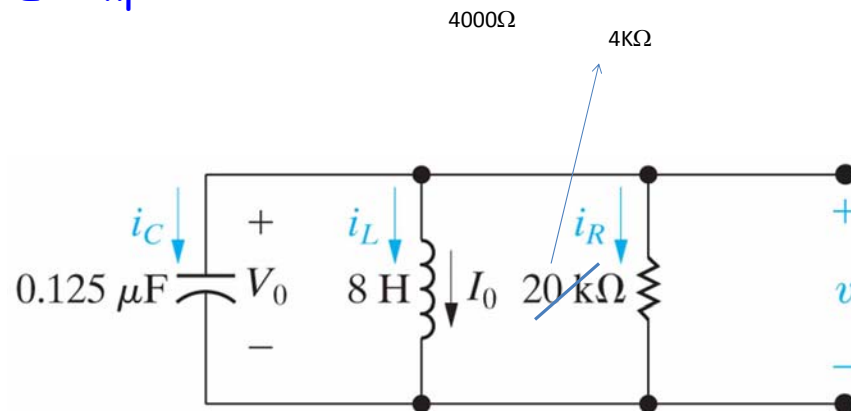
$$s_{1,2} = -\xi\omega_0 \pm \omega_0\sqrt{\xi^2 - 1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad \xi = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

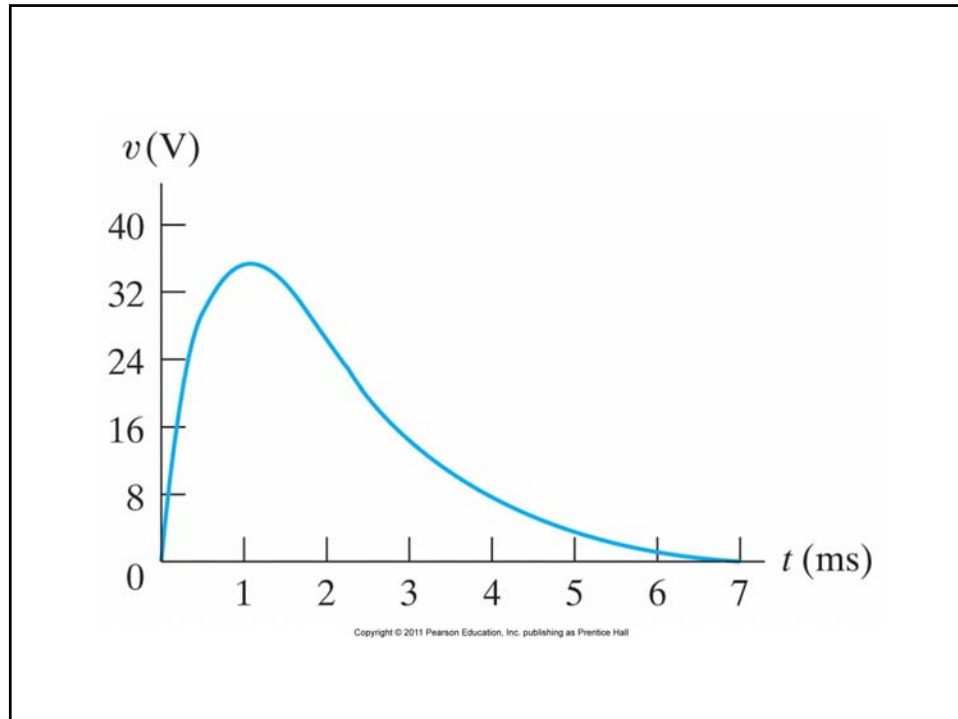
$$s_1 = s_2 = -\alpha$$

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Example



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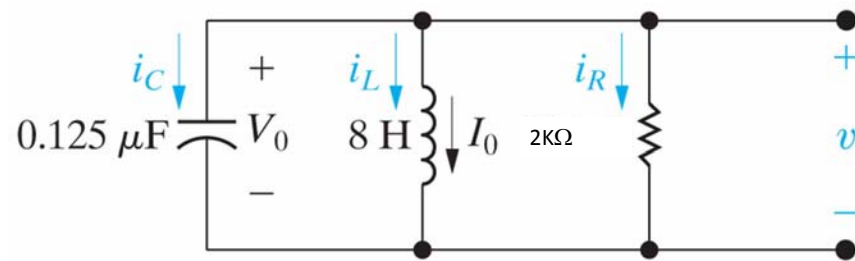


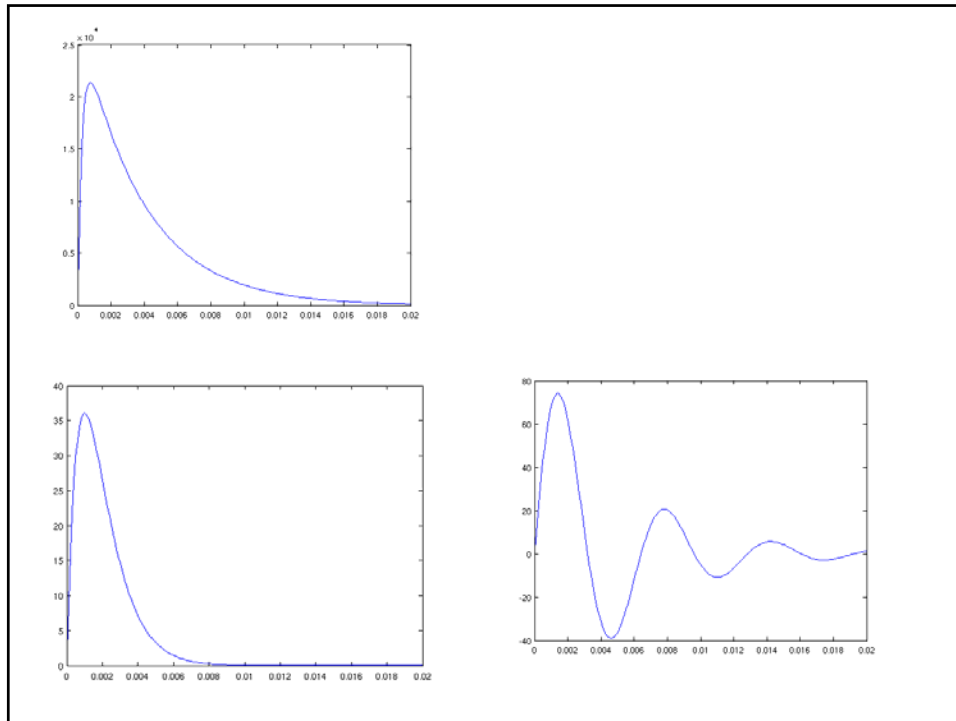
$$\xi = 2$$

$$\alpha = 2000, \quad \omega_0 = 1000$$

$$s_1 = -267.95$$

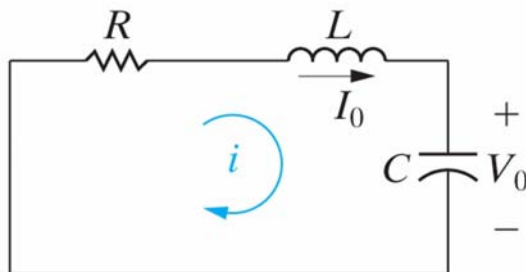
$$s_2 = 3732$$





Series RLC

- Solved in the lab manual
- Only difference is $\alpha=R/2L$



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