EECS 3213 Fall 2014

L13: Error Detection and Correction

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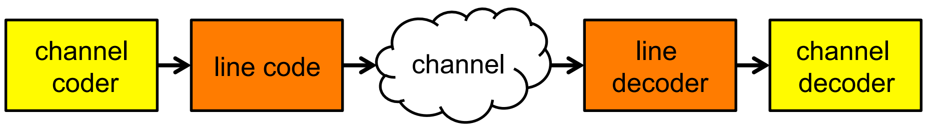
Outline

- Basic (channel) coding ideas
- Error detection (Backward error correction)
  - single parity
  - interleaved parity (2-D parity)
  - internet checksum
  - polynomial codes
- Effectiveness and Error Models
- (Forward) Error correction
  - cyclic codes, block codes, convolutional codes, iterative codes

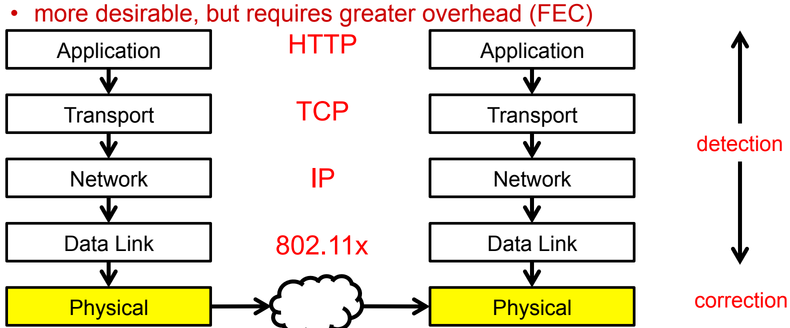
Types of Coding

- Some options
  - Line Coding
    - spectrum control
    - timing
    - basic error detection
  - Channel Coding
    - error detection
    - error correction
    - error prevention (combined detection & decoding)



Channel Coding

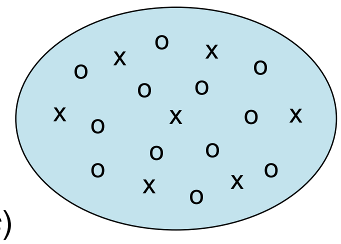
- Add in redundancy
- Two basic ideas, used in combination
  - **error detection**: recognizes an error in a frame (request re-send)
    - ARQ
  - **error correction**: finds error and corrects it (no need to re-send)
    - more desirable, but requires greater overhead (FEC)



Basic Ideas and Nomenclature

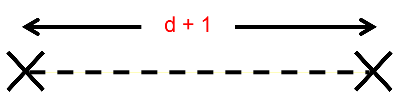
- Data consists of k bits
  - 2^k possible messages
- Add m redundant bits to this
- Codeword of n = m + k bits
  - $2^{m+k} = 2^n > 2^k$ possible strings
  - But only 2^k are valid!
  - (n,k) codes e.g.: (2,1)
- Thus coded messages (*codewords*) are separated in signal space
  - *Hamming distance*, d: # of bit positions that differ
- Code rate: r = k/n
  - ½, ¾

x = codeword
o = noncodewords

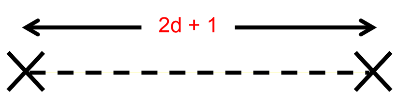


Detection and Correction Basic Example

- To detect d errors: need Hamming distance of d+1

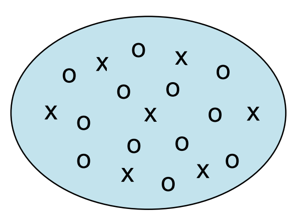


- To correct d errors: need Hamming distance of 2d+1

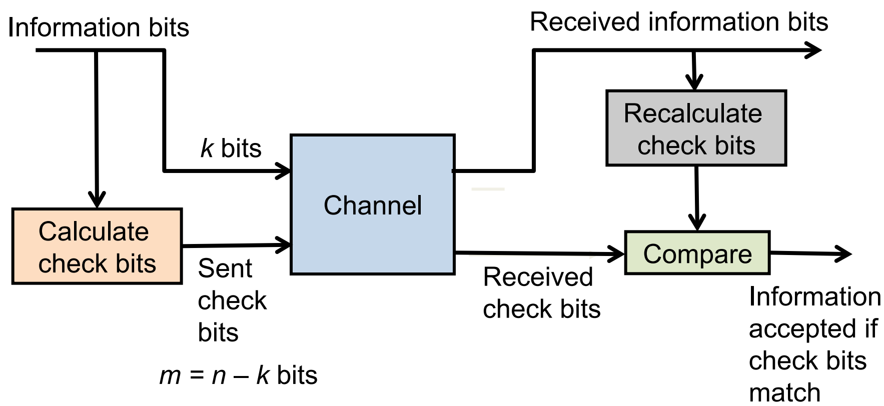


- e.g.
  - 0000000000
  - 0000011111
  - 1111100000
  - 1111111111

- $d_{min} = 5$
- detect up to 4 errors
- correct up to 2 errors
- only one at a time



System-Level Implementation



Simple Detection: Single Parity Code

- Information (7 bits): (0, 1, 0, 1, 1, 0, 0)
- Parity Bit: b_8 = 0 + 1 +0 + 1 +1 + 0 = 1
- Codeword (8 bits): (0, 1, 0, 1, 1, 0, 0, 1)
- If single error in bit 3 : (0, 1, 1, 1, 1, 0, 0, 1)
  - # of 1's =5, odd
  - Error detected
- If errors in bits 3 and 5: (0, 1, 1, 1, 0, 0, 0, 1)
  - # of 1's =4, even
  - Error not detected

Single Parity Check: Formally

- Append an overall parity check to k information bits
  - Info Bits: $b_1, b_2, b_3, \ldots, b_k$
  - Check Bit: $b_{k+1} = b_1 + b_2 + b_3 + \ldots + b_k$ modulo 2
  - Codeword: $(b_1, b_2, b_3, \ldots, b_{k,}, b_{k+1})$
- All codewords have even # of 1s
- Redundancy: Single parity check code adds 1 redundant bit per k information bits: overhead = $1/(k + 1)$
- Coverage
  - All error patterns that change an odd # of bits are detectable
  - All even-numbered patterns are undetectable
- Parity bit used in ASCII code

Effectiveness: Random Error Vector Model

- Effectiveness: Probability system fails to detect error
- Dependent on error model
  - Random Error Vector
  - Random Bit Error
  - Burst
- Random Error Vector
  - n-bit vector, ***e***, represents error pattern
  - $e_i = 0$ -> no error in position i
  - $e_i = 1$ -> an error in position i
- 2^n possible combinations
  - Assumes all possibilities equally likely
  - 50% chance of even number of errors
  - Therefore…????

What If Bit Errors are Random?

- Many transmission channels introduce bit errors at random, independent of each other, with probability p
- Some error patterns are more probable than others:
  - For example, if $p = 0.1$

$$P[10000000] = p(1-p)^7 = 0.0478$$
$$P[11000000] = p^2(1-p)^6 = 0.0053$$

- In any worthwhile channel $p < 0.5$, and so $p/(1-p) < 1$
- It follows (can you show this?) that patterns with 1 error are more likely than patterns with 2 errors and so forth
- What is the probability that an undetectable error pattern occurs?

Effectiveness: Random Bit Error Model

- Undetectable error pattern if even # of bit errors:

$$P[\text{error detection failure}] = P[\text{undetectable error pattern}]$$
$$= P[\text{error patterns with even number of 1s}]$$
$$= \binom{n}{2} p^2(1-p)^{n-2} + \binom{n}{4} p^4(1-p)^{n-4} + \ldots$$

- Example: Evaluate above for n = 32, p=10^{-3}

$$P[\text{undetectable error}] = \binom{32}{2} (10^{-3})^2 (1-10^{-3})^{30} + \binom{32}{4} (10^{-3})^4 (1-10^{-3})^{28}$$
$$\approx 496\ (10^{-6}) + 35960\ (10^{-12}) \approx 4.96\ (10^{-4})$$

- For this example, roughly 1 in 2000 error patterns is undetectable

Two-Dimensional Parity Check (Interleaving)

- More parity bits to improve coverage
- Arrange information as rows
- Add single parity bit to each row
- Add a final "parity" row
- Used in early error control systems

1	0	0	1	0	0
0	1	0	0	0	1
1	0	0	1	0	0
1	1	0	1	1	0
1	0	0	1	1	1

Last column consists of check bits for each row

Bottom row consists of check bit for each column

Error-Detecting Capability

1	0	0	1	0	0
0	0	0	0	0	1
1	0	0	1	0	0
1	1	0	1	1	0
1	0	0	1	1	1

One error

1	0	0	1	0	0
0	0	0	0	0	1
1	0	0	1	0	0
1	0	0	1	1	0
1	0	0	1	1	1

Two errors

- 1, 2, or 3 errors can always be detected
- Not all patterns >4 errors can be detected

1	0	0	1	0	0
0	0	0	1	0	1
1	0	0	1	0	0
1	0	0	1	1	0
1	0	0	1	1	1

Three errors

1	0	0	1	0	0
0	0	0	1	0	1
1	0	0	1	0	0
1	0	0	0	1	0
1	0	0	1	1	1

Four errors (undetectable)

Arrows indicate failed check bits

Other Error Detection Codes

- Many applications require very low error rate
- Need codes that detect the vast majority of errors
- Single parity check codes do not detect enough errors
- Two-dimensional codes require too many check bits
- The following error detecting codes used in practice:
  - Internet Check Sums
  - CRC Polynomial Codes

Internet Checksum

- Several Internet protocols (e.g. IP, TCP, UDP) use check bits in IP header to detect errors (or in the header and data for TCP/UDP)
- A checksum is calculated for header contents and included in a special field.
- Checksum recalculated at every router, so algorithm selected for ease of implementation in software
- Let header consist of L, 16-bit words, $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_{L-1}$
- The algorithm appends a 16-bit checksum $\mathbf{b}_L$

Checksum Calculation

The checksum $\mathbf{b}_L$ is calculated as follows:

- Treating each 16-bit word as an integer, find

  $\mathbf{x} = (\mathbf{b}_0 + \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{L-1})\text{modulo}(2^{16}-1)$

- The checksum is then given by:

  $\mathbf{b}_L = -\,\mathbf{x}\ \text{modulo}(2^{16}-1)$

Thus, the headers must satisfy the following pattern:

$$\mathbf{0} = (\mathbf{b}_0 + \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_{L-1} + \mathbf{b}_L)\text{modulo}(2^{16}-1)$$

- The checksum calculation is carried out in software using one's complement arithmetic

Internet Checksum Example

4	0100
5	0101
9	1001
18	10010
18 mod(2^4-1)	0010
= 3	+ 1
	0011

- Make checksum: -3
- 1100

- In the receiver

4	0100
5	0101
9	1001
-3	1100
15	11110
15 mod(2^4-1)	1110
= 0	+ 1
	1111

Polynomial Codes

- Polynomials instead of vectors for codewords
- Polynomial arithmetic instead of checksums
- Implemented using shift-register circuits
- Also called cyclic redundancy check (CRC) codes
- Most data communications standards use polynomial codes for error detection
- Polynomial codes also basis for powerful error-correction methods

General Idea

- Choose a special code: G (generator code, n=m+k bits)
- Shift information by m bits, ÷G, and find remainder, R

$$\frac{2^m I}{G} = Q \oplus \frac{R}{G}$$

- Make n=m+k bit codeword

$$B = 2^m I \oplus R$$

m-bit redundancy

- At receiver if no error:

$$\frac{B}{G} = \frac{2^m I \oplus R}{G}$$

$$= Q \oplus \frac{R}{G} \oplus \frac{R}{G} = Q$$

- At receiver if have error:

$$\frac{B \oplus E}{G} = \frac{2^m I \oplus R \oplus E}{G}$$

$$= C \oplus \frac{S}{G} \neq Q$$

Cyclic Error Correction

- We can do more than just detect…
- If have error:

$$\frac{B \oplus E}{G} = \frac{2^m I \oplus R \oplus E}{G} = C \oplus \frac{S}{G}$$

- But note:

$$\frac{B \oplus E}{G} = \underbrace{Q \oplus \frac{E}{G} = C \oplus \frac{S}{G}}$$

- *Rearranging*:

$$\frac{E}{G} = [Q \oplus C] \oplus \frac{S}{G}$$

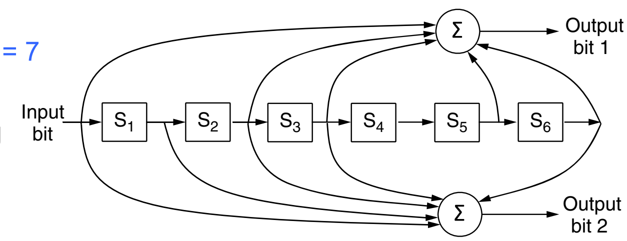
> - remainder (syndrome) depends only on the error (not on codeword B)
> - Syndrome can be used to identify error
> - As simple as LUT

Cyclic Code Types

- Cyclic codes are a type of block code
  - redundant bits are generated by some block of data (contrast with convolutional code)
- BCH codes are a specific example
  - (n,k,d)
  - (7,4,3): code rate = 4/7 = 0.571 (2 detect, 1 correct)
  - (15,5,7): code rate = 5/15 = 0.333 (6 detect, 3 correct)
- Reed-Solomon
  - operate on k-bit symbols (rather than individual bits)
  - and 2^k-1 symbols at a time (e.g. 8-bit symbol & 255 symbols total)
  - typical: (255,233,33), therefore can correct (33 – 1)/2 = 16 symbols
  - 8 x 16 = 128 bits in a 8 x 255 = 2040 bit sequence
  - very good for burst errors (DSL, cable, satellite, CDs)

Convolutional Codes

- Codes continuously
  - good for streaming, don't have to pause to collect blocks of bits
- Data is shifted through registers
  - output depends on present and past inputs (state-machine)
  - this redundancy achieves the necessary coding
- NASA convolutional code (Voyager)
  - (2,1), r = ½
  - constraint length = 7
  - GSM, 802.11
- Trellis decoding
  - Viterbi algorithm



[Tanenbaum11]

Recent Iterative Codes

- Turbo codes, 1993
  - two codes generated and interleaved
  - two decoders work iteratively to decode message
  - close to Shannon limit
- Low Density Parity Check, 1962 & 2003
  - block code
  - each output bit formed from only a fraction of input bits
  - iteratively re-assembled
  - rapidly being incorporated (no IP issues)
    - digital video, 10 Gbps ethernet, power line, latest 802.11