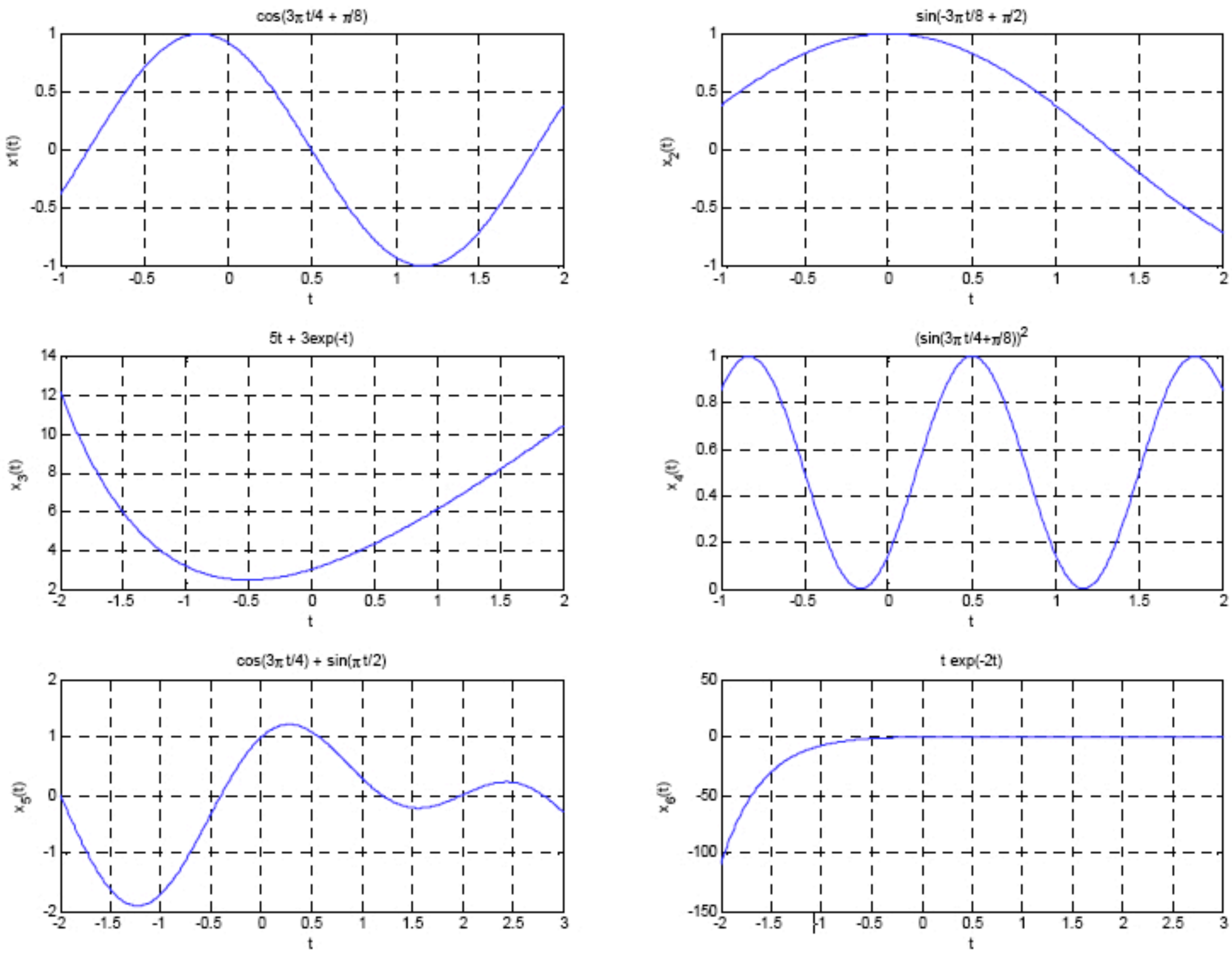
Instructors Solutions to Assignment 1

Problem 1.2:



Problem 1.5:

(i) All CT sinusoidal signals are periodic. The function $x1(t)$ can be simplified as follows:

$$x1(t) = \sin\left(-5\pi t/8 + \pi/2\right) = \sin\left(\pi/2 - 5\pi t/8\right) = \cos\left(5\pi t/8\right) = \cos\left(\omega_0 t\right), \omega_0 = 5\pi/8.$$

Therefore, $x1(t)$ is periodic with fundamental period

$$T_1 = \tfrac{2\pi}{\omega_0} = \tfrac{2\pi}{5\pi/8} = \tfrac{16}{5}.$$

(iv) All CT complex exponentials are periodic.

Therefore $x4(t) = \exp\left(j\left(5t + \pi/4\right)\right)$ is also periodic with fundamental period $T_4 = \frac{2\pi}{5}$.

(vii) $$x7(t) = \underbrace{1}_{\text{constant}} + \underbrace{\sin 20t}_{\substack{\textit{periodic} \\ T_1 = \frac{2\pi}{20} = \frac{\pi}{10}}} + \underbrace{\cos(30t + \pi/3)}_{\substack{\textit{periodic} \\ T_2 = \frac{2\pi}{30} = \frac{\pi}{15}}}$$

Since

$$\frac{T_1}{T_2} = \tfrac{\pi}{10} \times \tfrac{15}{\pi} = \tfrac{3}{2} = \text{rational number},$$

$x7(t)$ is periodic. The fundamental period of $x7(t)$ is $2T_1 = 3T_2 = \frac{\pi}{5}$.

Problem 1.10:

The CT signal $y(t) = A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$ is the sum of two sinusoids and may not be always periodic. It is periodic only when ω_1/ω_2 is a rational number. To consider the general case, where $y(t)$ is not necessarily periodic, we will use the general formula to evaluate the power in the signal.

$$P_y = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |y(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)|^2 \, dt$$

$$= \underbrace{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A_1^2 \sin^2(\omega_1 t + \phi_1) dt}_{=P_1} + \underbrace{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A_2^2 \sin^2(\omega_2 t + \phi_2) dt}_{=P_2} + \underbrace{\lim_{T\to\infty} \frac{2A_1A_2}{2T} \int_{-T}^{T} \sin(\omega_1 t + \phi_1)\sin(\omega_2 t + \phi_2) dt}_{=P_3}$$

The right hand side of the above equation includes three integrals. The first integral P_1 represents the power of a periodic signal $A_1 \sin(\omega_1 t + \phi_1)$. Based on Problem 1.9, the average power P_1 is given by $(A_1)^2/2$. Similarly, the secong integral $P_2 = (A_2)^2/2$. The third integral is evaluated by substituting

$$2\sin(\omega_1 t + \phi_1)\sin(\omega_1 t + \phi_1) = \cos(\omega_1 t + \phi_1 + \omega_2 t + \phi_2) - \cos(\omega_1 t + \phi_1 - \omega_2 t - \phi_2)$$

to get

$$P_3 = \lim_{T\to\infty} \frac{A_1A_2}{2T} \int_{-T}^{T} \cos[(\omega_1 + \omega_2)t + (\phi_1 + \phi_2)]dt + \lim_{T\to\infty} \frac{A_1A_2}{2T} \int_{-T}^{T} \cos[(\omega_1 - \omega_2)t + (\phi_1 - \phi_2)]dt .$$

Case $\omega_1 \neq \omega_2$: In such a case, both integrals result in finite values giving

$$P_3 = \lim_{T\to\infty} \frac{A_1A_2}{2T} \times (\text{finite value} \# 1) + \lim_{T\to\infty} \frac{A_1A_2}{2T} (\text{finite value} \# 2) = 0 .$$

Case $\omega_1 = \omega_2$: In such a case, we obtain

$$P_3 = \lim_{T\to\infty} \frac{A_1A_2}{2T} \times (\text{finite value \# 1}) + \lim_{T\to\infty} \frac{A_1A_2}{2T} \int_{-T}^{T} \cos[(\phi_1 - \phi_2)]dt$$

$$= 0 + \lim_{T\to\infty} \frac{A_1A_2}{2T} 2T \cos[(\phi_1 - \phi_2)] = A_1A_2 \cos[(\phi_1 - \phi_2)].$$

Combining the above results, we obtain

$$P_y = \begin{cases} \frac{A_1^2}{2} + \frac{A_2^2}{2} & \omega_1 \neq \omega_2 \\ \frac{A_1^2}{2} + \frac{A_2^2}{2} + A_1A_2 \cos(\phi_1 - \phi_2) & \omega_1 = \omega_2 . \end{cases}$$

Problem 1.22:

(i) $$\int_{-\infty}^{\infty} (t-1)\delta(t-5)dt = \int_{-\infty}^{\infty} 4\delta(t-5)dt = 4\int_{-\infty}^{\infty} \delta(t-5)dt = 4 .$$

(ii) $$\int_{-\infty}^{6} (t-1)\delta(t-5)dt = \int_{-\infty}^{6} 4\delta(t-5)dt = 4\int_{-\infty}^{6} \delta(t-5)dt = 4 .$$

(iii) $$\int_6^\infty (t-1)\delta(t-5)dt = \int_6^\infty 4\,\delta(t-5)dt = 4\int_6^\infty \delta(t-5)dt = 0\,.$$

(iv) $$\int_{-\infty}^{\infty} (2t/3-5)\delta(3t/4-5/6)dt = \int_{-\infty}^{\infty} \left(\tfrac{2}{3}t-5\right)\delta\left(\tfrac{3}{4}(t-\tfrac{10}{9})\right)dt = \tfrac{4}{3}\int_{-\infty}^{\infty} \left(\tfrac{2}{3}t-5\right)\delta\left(t-\tfrac{10}{9}\right)dt$$

which simplifies to

$$= \tfrac{4}{3}\int_{-\infty}^{\infty} \Bigg(\underbrace{\tfrac{2}{3}\times\tfrac{10}{9}-5}_{\approx -115/27} \Bigg)\delta\left(t-\tfrac{10}{9}\right)dt = \tfrac{-460}{81}\int_{-\infty}^{\infty} \delta\left(t-\tfrac{10}{9}\right)dt = \tfrac{-460}{81}\,.$$

(v) $$\int_{-\infty}^{\infty} \exp(t-1)\sin\left(\pi(t+5)/4\right)\delta(1-t)dt = \int_{-\infty}^{\infty} \exp(t-1)\sin\left(\pi(t+5)/4\right)\delta(t-1)dt$$

which simplifies to

$$= \int_{-\infty}^{\infty} \exp(0)\sin(\pi 6/4)\delta(t-1)dt = \sin(\pi 6/4)\int_{-\infty}^{\infty} \delta(t-1)dt = \sin(3\pi/2) = -1\,.$$

(vi) $$\int_{-\infty}^{\infty} \left[\sin(3\pi t/4) + e^{-2t+1}\right]\delta(-(t+1))dt = \int_{-\infty}^{\infty} \left[\sin(3\pi t/4) + e^{-2t+1}\right]\delta(t+1)dt = \left[\sin(3\pi t/4) + e^{-2t+1}\right]\Big|_{t=-1}$$

which simplifies to

$$= \sin(-3\pi/4) + e^3 = e^3 - \sin(3\pi/4) = e^3 - \tfrac{1}{\sqrt{2}}\,.$$

(vii) $$\int_{-\infty}^{\infty} \left[u(t-6) - u(t-10)\right]\sin(3\pi t/4)\delta(t-5)dt = \left[u(t-6) - u(t-10)\right]\sin(3\pi t/4)\Big|_{t=5}$$

which simplifies to

$$= \left[u(5-6) - u(5-10)\right]\sin(3\pi 5/4) = [0-0]\sin(15\pi/4) = 0\,.$$

(viii) By noting that only the impulses located at $t = -20$ ($m = -4$), $t = -15$ ($m = -3$), $t = -10$ ($m = -2$), $t = -5$ ($m = -1$), $t = 0$ ($m = 0$), $t = 5$ ($m = 1$), $t = 10$ ($m = 2$), $t = 15$ ($m = 3$), and $t = 20$ ($m = 4$) lie within the integration range of ($-21 \le t \le 21$), the integral reduces to

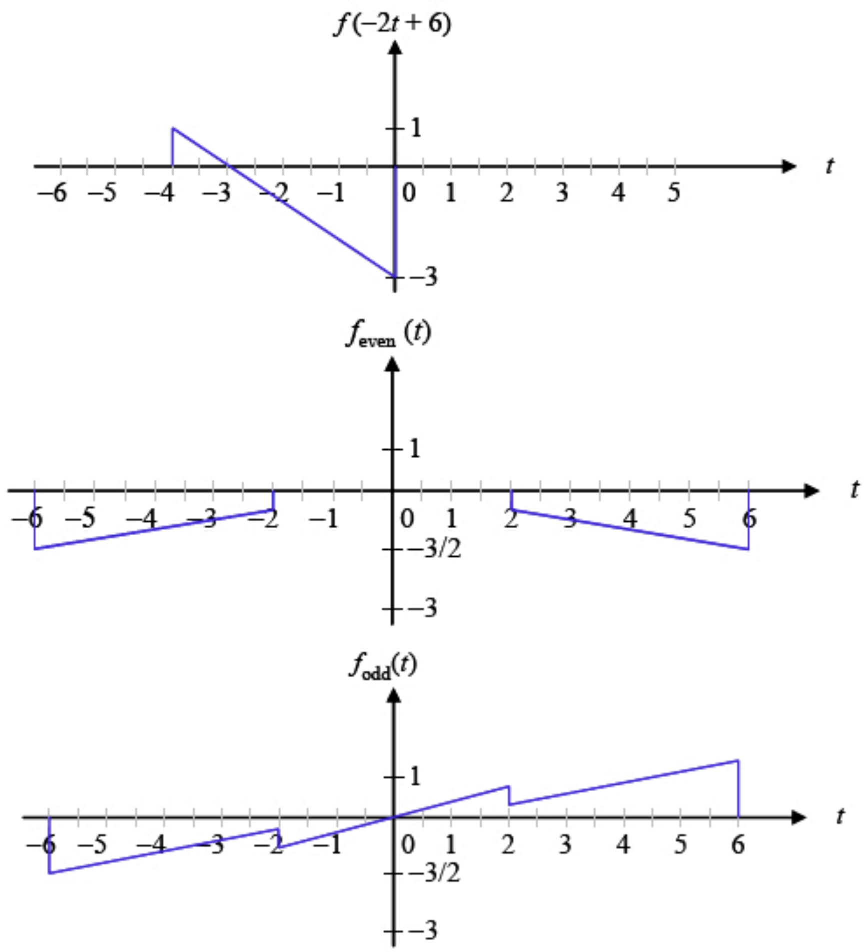
$$I = \int_{-21}^{21} \left(\sum_{m=-\infty}^{\infty} t\delta(t-5m) \right) dt = \int_{-21}^{21} \left(\sum_{m=-4}^{4} t\delta(t-5m) \right) dt\,.$$

Changing the order of summation and integration, we obtain

$$I = \sum_{m=-4}^{4} \int_{-21}^{21} t\delta(t-5m)dt = \sum_{m=-4}^{4} 5m = 5(-4-3-2-1+0+1+2+3+4) = 0.$$

▮

Problem 1.26



Problem 2.9

(i) $y(t) = x(t-2)$

(a) Linearity: Since

$$x_1(t) \to x_1(t-2) = y_1(t)$$
$$x_2(t) \to x_2(t-2) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(t-2) + \beta x_2(t-2) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t) \to x_1(t-2) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(t-2) = x_1(t-T-2) = y_2(t)$$

and $y_1(t-T) = x_1(t-T-2) = y_2(t)$, the system is time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = |x(t-2)| \le M$$

is also bounded proving that the system is BIBO stable.

(d) Causality: Since the output depends only on the past input and does not depend on the future values of the input, therefore, the system is causal.

(ii) $y(t) = x(2t-5)$

(a) Linearity: Since

$$x_1(t) \to x_1(2t-5) = y_1(t)$$
$$x_2(t) \to x_2(2t-5) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(2t-5) + \beta x_2(2t-5) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t) \to x_1(2t-5) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(2t-5) = y_2(t)$$

which implies that

$$x_2(t) = x_1(t-T) \to y_2(t) = x_2(2t-5)\big|_{x_2(t)=x_1(t-T)} = x_1(2t-5-T).$$

On the other hand,

$$y_1(t-T) = x_1(2t-5)\big|_{t=t-T} = x_1(2(t-T)-5) = x_1(2t-2T-5),$$

and $y_1(t-T) \ne y_2(t)$. Therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \le M$. Then, the output

$$|y(t)| = |x(2t-5)| \le M$$

is also bounded proving that the system is BIBO stable.

(d) Causality: For ($t > 5$), the output depends on the future values of the input, therefore, the system is NOT causal.

(iii) $y(t) = x(2t) - 5$

(a) Linearity: Since

$$x_1(t) \to x_1(2t) - 5 = y_1(t)$$
$$x_2(t) \to x_2(2t) - 5 = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha x_1(2t) + \beta x_2(2t) - 5 \ne \alpha y_1(t) + \beta y_2(t)$$

because $\alpha y_1(t) + \beta y_2(t) = \alpha x_1(2t) + \beta x_2(2t) - 5(\alpha + \beta)$. Therefore, the system is NOT linear.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t) \to x_1(2t) - 5 = y_1(t)$$
$$x_2(t) = x_1(t-T) \to x_2(2t) - 5 = y_2(t)$$

which implies that

$$x_2(t) = x_1(t-T) \to y_2(t) = x_2(2t) - 5\Big|_{x_2(t)=x_1(t-T)} = x_1(2t-T) - 5\,.$$

On the other hand,

$$y_1(t-T) = x_1(2t)\Big|_{t=t-T} - 5 = x_1(2(t-T)) - 5 = x_1(2t-2T) - 5\,.$$

Clearly, $y_1(t-T) \neq y_2(t)$, therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = |x(2t) - 5| \leq |x(2t)| + 5 \leq M + 5$$

is also bounded proving that the system is BIBO stable.

(d) Causality: For (t > 0), the system requires future values of the input to calculate the current value of the input. Therefore, the system is NOT causal.

(iv) $y(t) = tx(t+10)$

(a) Linearity: Since

$$x_1(t) \to tx_1(t+10) = y_1(t)$$
$$x_2(t) \to tx_2(t+10) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to \alpha t x_1(t+10) + \beta t x_2(t+10) = \alpha y_1(t) + \beta y_2(t)$$

therefore, the system is a linear system.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t-T)$, the outputs are given by

$$x_1(t) \to tx_1(t+10) = y_1(t)$$
$$x_2(t) = x_1(t-T) \to tx_2(t+10) = tx_1(t-T+10) = y_2(t)$$

We also note that $$y_1(t-T) = (t-T)x_1(t-T+10) \neq y_2(t),$$

therefore, the system is NOT time invariant.

(c) Stability: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = |tx(t+10)| = |t||x(t+10)| \leq M|t|$$

is unbounded as $t \to \infty$. Therefore, the system is NOT BIBO stable.

(d) Causality: Since the output depends on the future values of the input, and therefore the system is NOT causal.

(v) $y(t) = 2u(x(t)) = \begin{cases} 2 & x(t) \geq 0 \\ 0 & x(t) < 0 \end{cases}$

(a) Linearity: Since

$$x_1(t) \to 2u(x_1(t)) = y_1(t)$$
$$x_2(t) \to 2u(x_2(t)) = y_2(t)$$
$$\alpha x_1(t) + \beta x_2(t) \to 2u(\alpha x_1(t) + \beta x_2(t)) = y(t)$$

and $\alpha y_1(t)+\beta y_2(t)=2\alpha u\big(x_1(t)\big)+2\beta u\big(x_2(t)\big)\neq y(t)$. Therefore, the system is NOT linear.

Also, we note that

$$y_2(t)-y_1(t)=2u\big(x_2(t)\big)-2u\big(x_1(t)\big)\neq \lambda\left[x_2(t)-x_1(t)\right].$$

Therefore, the system is NOT an incrementally linear system either.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t)\rightarrow 2u\big(x_1(t)\big)=y_1(t)$$
$$x_2(t)=x_1(t-T)\rightarrow 2u\big(x_2(t)\big)=2u\big(x_1(t-T)\big)=y_2(t)$$

We note that $y_1(t-T)=2u\big(x_1(t-T)\big)=y_2(t)$, therefore, the system is time invariant.

(c) Stability: Since $|y(t)| \leq 2$, therefore, the system is BIBO stable.

(d) Causality: The output at any time instant does not depend on future value of the input. The system is, therefore, causal.

(vi) $$y(t)=\begin{cases}0 & \text{t<0}\\ x(t)-x(t-5) & \text{t}\geq 0\end{cases} \qquad =\left[x(t)-x(t-5)\right]u(t)$$

(a) Linearity: Since

$$x_1(t)\rightarrow\left[x_1(t)-x_1(t-5)\right]u(t)=y_1(t)$$
$$x_2(t)\rightarrow\left[x_2(t)-x_2(t-5)\right]u(t)=y_2(t)$$
$$\alpha x_1(t)+\beta x_2(t)\rightarrow\left[\{\alpha x_1(t)+\beta x_2(t)\}-\{\alpha x_1(t-5)+\beta x_2(t-5)\}\right]u(t)=y(t)$$
$$=\alpha\left[x_1(t)-x_1(t-5)\right]u(t)+\beta\left[x_1(t)-x_1(t-5)\right]u(t)$$
$$=\alpha y_1(t)+\beta y_2(t),$$

the system is linear.

(b) Time Invariance: For inputs $x_1(t)$ and $x_2(t) = x_1(t - T)$, the outputs are given by

$$x_1(t)\rightarrow\left[x_1(t)-x_1(t-5)\right]u(t)=y_1(t)$$
$$x_2(t)=x_1(t-T)\rightarrow\left[x_2(t)-x_2(t-5)\right]u(t)=y_2(t)$$

We note that $y_1(t-T)\neq y_2(t)$ since

$$y_1(t-T)=y_1(t)\Big|_{t=t-T}=\left[x_1(t)-x_1(t-5)\right]u(t)\Big|_{t=t-T}=\left[x_1(t-T)-x_1(t-T-5)\right]u(t-T)$$

and $$y_2(t)=\left[x_2(t)-x_2(t-5)\right]u(t)==\left[x_1(t-T)-x_1(t-T-5)\right]u(t).$$

Therefore, the system is NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$\left|y(t)\right|=\left|x(t)-x(t-5)\right|\leq\left|x(t)\right|+\left|x(t-5)\right|\leq 2M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: The output does not depend on the future values of the input, therefore, the system is causal.

(vii) $y(t) = 7x^2(t) + 5x(t) + 3$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$y_1(t) = 7x_1^2(t) + 5x_1(t) + 3$$

For $x_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$y_2(t) = 7x_2^2(t) + 5x_2(t) + 3.$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = 7(\alpha x_1(t) + \beta x_2(t))^2 + 5(\alpha x_1(t) + \beta x_2(t)) + 3,$$

or, $$y_3(t) = \alpha\underbrace{\left(7x_1^2(t) + 5x_1(t) + 3\right)}_{y_1(t)} + \beta\underbrace{\left(7x_1^2(t) + 5x_2(t) + 3\right)}_{y_2(t)} + 14\alpha\beta x_1(t)x_2(t) + 3(1 - \alpha - \beta)$$

The above result implies that

$$y_3(t) \neq \alpha y_1(t) + \beta y_2(t),$$

And hence the system is not linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$x_1(t) \rightarrow 7x_1^2(t) + 5x_1(t) + 3 = y_1(t)$$
$$x_2(t) = x_1(t-T) \rightarrow 7x_2^2(t) + 5x_2(t) + 3 = y_2(t).$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = 7x_1^2(t-T) + 5x_2(t-T) + 3.$$

We also note that $$y_1(t-T) = 7x_1^2(t-T) + 5x_2(t-T) + 3,$$

implying that $y_1(t-T) = y_2(t)$. The system is, therefore, time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = \left|7x^2(t) + 5x(t) + 3\right| \leq 7|x(t)||x(t)| + 5|x(t)| + 3 \leq 7M^2 + 5M + 3$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: The output $y(t)$ at $t = t_0$ requires only one value of the input $y(t)$ at ($t = t_0$). Therefore, the system is causal.

(viii) $y(t) = \text{sgn}(x(t))$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t) = \text{sgn}(x_1(t))$.

For $x_2(t)$ applied as the input, the output $y_2(t) = \text{sgn}(x_2(t))$.

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = \text{sgn}(\alpha x_1(t) + \beta x_2(t)) \neq \alpha\,\text{sgn}(x_1(t)) + \beta\,\text{sgn}(x_2(t)).$$

The above result implies that

$$y_3(t) \neq \alpha y_1(t) + \beta y_2(t),$$

And hence the system is not linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \rightarrow \mathrm{sgn}(x_1(t)) = y_1(t)$$
$$x_2(t) = x_1(t-T) \rightarrow \mathrm{sgn}(x_2(t)) = y_2(t)$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \mathrm{sgn}(x_1(t-T)).$$

We also note that $$y_1(t-T) = \mathrm{sgn}(x_1(t-T)),$$

implying that $y_1(t-T) = y_2(t)$. The system is, therefore, time invariant.

(c) Stablity: The system is stable as the output is always bounded between the values of −1 and 1.

(d) Causality: The output $y(t)$ at ($t = t_0$) requires only one value of the input $x(t)$ at ($t = t_0$), therefore, the system is causal.

(ix) $y(t) = \int_{-t_0}^{t_0} x(\lambda)d\lambda + 2x(t)$

(a) Linearity: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$y_1(t) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t),$$

$$y_2(t) = \int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t).$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$y_3(t) = \int_{-t_0}^{t_0} \left(\alpha x_1(\lambda) + \beta x_2(\lambda)\right) d\lambda + 2\left(\alpha x_1(t) + \beta x_2(t)\right)$$

$$= \alpha \Bigg[\underbrace{\int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t)}_{y_1(t)} \Bigg] + \beta \Bigg[\underbrace{\int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t)}_{y_2(t)} \Bigg],$$

$$= \alpha y_1(t) + \beta y_2(t)$$

Therefore, the system is linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \rightarrow y_1(t) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t)$$

$$x_2(t) = x_1(t-T) \rightarrow y_2(t) = \int_{-t_0}^{t_0} x_2(\lambda)d\lambda + 2x_2(t),$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \int_{-t_0}^{t_0} x_1(\lambda - T)d\lambda + 2x_1(t-T).$$

By substituting $\lambda' = \lambda - T$, we get $y_2(t) = \int_{-t_0-T}^{t_0-T} x_1(\lambda')d\lambda' + 2x_1(t-T)$.

We also note that $$y_1(t-T) = \int_{-t_0}^{t_0} x_1(\lambda)d\lambda + 2x_1(t-T),$$

implying that $y_1(t-T) \neq y_2(t)$. The system is, therefore, NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = \left| \int_{-t_0}^{t_0} x(\lambda)d\lambda + 2x(t) \right| \leq \int_{-t_0}^{t_0} |x(\lambda)|d\lambda + 2|x(t)| \leq 2Mt_0 + 2M$$

is also bounded. Therefore, the system is BIBO stable.

(d) Causality: To solve the integral, the output $y(t)$ always requires the values of the input $x(t)$ within the range ($-t_0 \leq t \leq t_0$) no matter when $y(t)$ (even for $t < -t_0$) is being determined. Therefore, the system is NOT causal.

(x) $$y(t) = \int_{-\infty}^{t_0} x(\lambda)d\lambda + \frac{dx}{dt}$$

(a) Linearity: For $x_1(t)$ and $x_2(t)$ applied as the inputs, the outputs are given by

$$y_1(t) = \int_{-\infty}^{t_0} x_1(\lambda)d\lambda + \frac{dx_1}{dt},$$

$$y_2(t) = \int_{-\infty}^{t_0} x_2(\lambda)d\lambda + \frac{dx_2}{dt}.$$

For $x_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\begin{aligned} y_3(t) &= \int_{-\infty}^{t_0} \left(\alpha x_1(\lambda) + \beta x_2(\lambda)\right)d\lambda + \frac{d\left(\alpha x_1(t) + \beta x_2(t)\right)}{dt} \\ &= \alpha \underbrace{\left[\int_{-\infty}^{t_0} x_1(\lambda)d\lambda + \frac{d\left(x_1(t)\right)}{dt}\right]}_{y_1(t)} + \beta \underbrace{\left[\int_{-\infty}^{t_0} x_2(\lambda)d\lambda + \frac{d\left(x_2(t)\right)}{dt}\right]}_{y_2(t)}, \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Therefore, the system is linear.

(b) Time Invariance: For $x_1(t)$ and $x_2(t) = x_1(t-T)$ applied as the inputs, the outputs are given by

$$x_1(t) \to y_1(t) = \int_{-\infty}^{t_0} x_1(\lambda) d\lambda + \frac{dx_1}{dt}$$

$$x_2(t) = x_1(t-T) \to y_2(t) = \int_{-\infty}^{t_0} x_2(\lambda) d\lambda + \frac{dx_2}{dt}.$$

Substituting $x_2(t) = x_1(t-T)$ we obtain,

$$y_2(t) = \int_{-\infty}^{t_0} x_1(\lambda - T) d\lambda + \frac{dx_1(t-T)}{dt}.$$

By substituting $\lambda' = \lambda - T$, we get $y_2(t) = \int_{-\infty}^{t_0 - T} x_1(\lambda') d\lambda' + \frac{dx_1(t-T)}{dt}$.

We also note that $$y_1(t-T) = \int_{-\infty}^{t_0} x_1(\lambda) d\lambda + \frac{dx_1(t-T)}{dt},$$

implying that $y_1(t-T) \neq y_2(t)$. The system is, therefore, NOT time invariant.

(c) Stablity: Assume that the input is bounded $|x(t)| \leq M$. Then, the output

$$|y(t)| = \left| \int_{-\infty}^{t_0} x(\lambda) d\lambda + 2\frac{dx(t)}{dt} \right| \leq \int_{-\infty}^{t_0} |x(\lambda)| d\lambda + 2\left|\frac{dx(t)}{dt}\right|$$

is unbounded because of the integral which integrates $x(t)$ from $(-\infty \leq t \leq t_0)$. Therefore, the system is NOT stable.

(d) Causality: To solve the integral, the output $y(t)$ always requires only the values of the input $x(t)$ within the range $(-\infty \leq t \leq t_0)$ no matter when $y(t)$ (even for $t < -t_0$) is being determined. Therefore, the system is NOT causal.

(xi) $$\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y(t) = \frac{d^2x}{dt^2} + 2x(t) + 1$$

(a) Linearity: For $x_1(t)$ applied as the input, the output $y_1(t)$ is given by

$$\frac{d^4y_1}{dt^4} + 3\frac{d^3y_1}{dt^3} + 5\frac{d^2y_1}{dt^2} + 3\frac{dy_1}{dt} + y_1(t) = \frac{d^2x_1}{dt^2} + 2x_1(t) + 1. \quad \text{(S2.9.1)}$$

For $x_2(t)$ applied as the input, the output $y_2(t)$ is given by

$$\frac{d^4y_2}{dt^4} + 3\frac{d^3y_2}{dt^3} + 5\frac{d^2y_2}{dt^2} + 3\frac{dy_2}{dt} + y_2(t) = \frac{d^2x_2}{dt^2} + 2x_2(t) + 1. \quad \text{(S2.9.2)}$$

For $y_3(t) = \alpha\, x_1(t) + \beta\, x_2(t)$ applied as the input, the output $y_3(t)$ is given by

$$\frac{d^4y_3}{dt^4}+3\frac{d^3y_3}{dt^3}+5\frac{d^2y_3}{dt^2}+3\frac{dy_3}{dt}+y_2(t)=\frac{d^2\big(\alpha x_1(t)+\beta x_2(t)\big)}{dt^2}+2\big(\alpha y_1(t)+\beta y_2(t)\big)+1\,,$$

or, $$\frac{d^4y_3}{dt^4}+3\frac{d^3y_3}{dt^3}+5\frac{d^2y_3}{dt^2}+3\frac{dy_3}{dt}+y_2(t)=\alpha\underbrace{\left[\frac{d^2x_1}{dt^2}+2x_1(t)+1\right]}_{\text{Term I}}+\beta\underbrace{\left[\frac{d^2x_2}{dt^2}+2x_2(t)+1\right]}_{\text{Term II}}+[1-\alpha-\beta].$$

Substituting the values of the derivative terms (Terms I and II) from Eqs. (S2.9.1) ans (2.9.2), we obtain

$$\frac{d^4y_3}{dt^4}+3\frac{d^3y_3}{dt^3}+5\frac{d^2y_3}{dt^2}+3\frac{dy_3}{dt}+y_2(t)=\alpha\underbrace{\left[\frac{d^4y_1}{dt^4}+3\frac{d^3y_1}{dt^3}+5\frac{d^2y_1}{dt^2}+3\frac{dy_1}{dt}+y_1(t)\right]}_{\text{Term I}}$$
$$+\beta\underbrace{\left[\frac{d^4y_2}{dt^4}+3\frac{d^3y_2}{dt^3}+5\frac{d^2y_2}{dt^2}+3\frac{dy_2}{dt}+y_2(t)\right]}_{\text{Term II}}+[1-\alpha-\beta]$$

which implies that $y_3(t)\neq\alpha y_1(t)+\beta y_2(t)$.

The system is, therefore, NOT linear. Note that the dc term of (+ 1) on the right hand side of the differential equation contributes to the nonlinearity of the system

(b) Time-invariance: The system is time-invariant. The proof is similar to Problem 2.1.

(b) Time-invariance: For $x_1(t)$ and $x_2(t)=x_1(t-T)$ applied as the inputs, the outputs are given by

$$\frac{d^4y_1}{dt^4}+3\frac{d^3y_1}{dt^3}+5\frac{d^2y_1}{dt^2}+3\frac{dy_1}{dt}+y_1(t)=\frac{d^2x_1}{dt^2}+2x_1(t)+1 \qquad \text{(S2.9.3)}$$

$$\frac{d^4y_2}{dt^4}+3\frac{d^3y_2}{dt^3}+5\frac{d^2y_2}{dt^2}+3\frac{dy_2}{dt}+y_2(t)=\frac{d^2x_2}{dt^2}+2x_2(t)+1\,.$$

Substituting $x_2(t)=x_1(t-T)$ we obtain,

$$\frac{d^4y_2}{dt^4}+3\frac{d^3y_2}{dt^3}+5\frac{d^2y_2}{dt^2}+3\frac{dy_2}{dt}+y_2(t)=\frac{d^2x_1(t-T)}{dt^2}+2x_1(t-T)+1\,. \qquad \text{(S2.9.4)}$$

Substituting $\tau=t+T$ (which implies that $dt=d\tau$) in Eq. (S2.9.3), we obtain

$$\frac{d^4y_1(\tau-T)}{d\tau^4}+3\frac{d^3y_1(\tau-T)}{d\tau^3}+5\frac{d^2y_1(\tau-T)}{d\tau^2}+3\frac{dy_1(\tau-T)}{d\tau}+y_1(\tau-T)=\frac{d^2x_1(\tau-T)}{d\tau^2}+2x_1(\tau-T)+1\,.$$

Or, $$\frac{d^4y_1(t-T)}{dt^4}+3\frac{d^3y_1(t-T)}{dt^3}+5\frac{d^2y_1(t-T)}{dt^2}+3\frac{dy_1(t-T)}{dt}+y_1(t-T)=\frac{d^2x_1(t-T)}{dt^2}+2x_1(t-T)+1\,.$$

Comparing with Eq. (S2.9.4), we obtain

$$y_2(t)=y_1(t-T)\,,$$

proving that the system is time-invariant.

(c) Stablity: The system is BIBO stable since a bounded input will always produce a bounded output.

(d) Causality: Express Eq. (S2.2) as follows:

$$y(t) = -3\int_{-\infty}^{t} y(\alpha)d\alpha - 5\int_{-\infty}^{t}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - 3\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta - \int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} y(\alpha)d\alpha d\tau d\theta d\phi$$
$$+5\int_{-\infty}^{t}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau + 2\int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta + \int_{-\infty}^{t}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} d\alpha d\tau d\theta d\phi$$

The output $y(t)$ at $t = t_0$ is given by

$$y(t)\big|_{t=t_0} = -3\int_{-\infty}^{t_0} y(\alpha)d\alpha - 5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau} y(\alpha)d\alpha d\tau - 3\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta - \int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} y(\alpha)d\alpha d\tau d\theta d\phi$$
$$+5\int_{-\infty}^{t_0}\int_{-\infty}^{\tau} x(\alpha)d\alpha d\tau + 2\int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta} y(\alpha)d\alpha d\tau d\theta + \int_{-\infty}^{t_0}\int_{-\infty}^{\tau}\int_{-\infty}^{\theta}\int_{-\infty}^{\phi} d\alpha d\tau d\theta d\phi$$

The system is causal since only the past values of the input $x(t)$, for $-\infty \le t \le t_0$, are needed to calculate the output $y(t)$ at $t = t_0$. $\blacksquare$

Problem 2.13

(i) The system is invertible with the inverse system given by

$$x(t) = \tfrac{1}{3}y(t-2).$$

(ii) To calculate the inverse system, we differentiate the integral to get

$$\frac{dy(t)}{dt} = x(t-10).$$

The inverse system is obtained through two steps. Step 1 compute $z(t) = dy/dt$, while Step 2 computes $x(t)$ from the relationship $x(t) = z(t + 10)$.

(iii) The system $y(t) = |x(t)|$ is not invertible as $x(t) = \pm a$ produces the same output $y(t) = a$.

(iv) If $y(t)$ is differentiable then $x(t)$ can always be calculated uniquely from the expression

$$x(t) = \frac{dy(t)}{dt} + y(t)$$

and the system is invertible. However, if $y(t)$ is not differentiable (for example, it contains a discontinuity), then $x(t)$ cannot always be calculated uniquely and the system is not invertible.

(v) System represented by $y(t) = \cos(2\pi x(t))$ is not invertible as different values of $x(t) = (\theta + 2m\pi)$, where m is an integer, produce the same output.