Instructors Solutions to Assignment 2

Problem 3.2

(i) $\ddot{y}(t)+4\dot{y}(t)+8y(t)=\dot{x}(t)+x(t)$ with $x(t)=e^{-4t}u(t), y(0)=0,$ and $\dot{y}(0)=0.$

(a) Particular solution: The particular solution for input $x(t) = \exp(-4t)u(t)$ is of the form

$$y_p(t) = Ke^{-4t}u(t).$$

Substituting the particular solution in the differential equation for system (i) and solving the resulting equation gives $K = -3/8$.

(b) Homogeneous solution: The characteristic equation of the LTIC system (i) is

$$s^2+4s+8=0,$$

which has roots at $s = -2 \pm j2$. The zero-input response is given by

$$y_h(t) = Ae^{-2t}\cos(2t) + Be^{-2t}\sin(2t)$$

for $t \geq 0$, with A and B being constants.

(c) Overall response of the system: The overall response of the system is obtained by summing up the above two responses, and use initial conditions to derive A and B, and it is given by

$$y(t)=\tfrac{3}{8}\left(e^{-2t}\cos(2t)-e^{-2t}\sin(2t)-e^{-4t}\right)u(t).$$

(iii) $\ddot{y}(t)+2\dot{y}(t)+y(t)=\ddot{x}(t)$ with $x(t)=\left[\cos(t)+\sin(2t)\right]u(t), y(0)=3,$ and $\dot{y}(0)=1.$

(a) Particular solution: The particular solution for input $x(t) = [\cos(t) + \sin(t)]u(t)$ is of the form

$$y_p(t) = K_1\cos(t)+K_2\sin(t)+K_3\cos(2t)+K_4\sin(2t).$$

Substituting the particular solution in the differential equation for system (iii) and solving the resulting equation gives

$$\begin{aligned}&\left(-K_1\cos(t)-K_2\sin(t)-4K_3\cos(2t)-4K_4\sin(2t)\right)+2\left(-K_1\sin(t)+K_2\cos(t)-2K_3\sin(2t)\right.\\&\left.+2K_4\cos(2t)\right)+1\left(K_1\cos(t)+K_2\sin(t)+K_3\cos(2t)+K_4\sin(2t)\right)=-\cos(t)-4\sin(2t)\end{aligned}$$

Collecting the coefficients of the cosine and sine terms, we get

$$\begin{aligned}&\left(-K_1+2K_2+K_1+1\right)\cos(t)+\left(-K_2-2K_1+K_2\right)\sin(t)+\\&\left(-4K_3+4K_4+K_3\right)\cos(2t)+\left(-4K_4-4K_3+K_4+4\right)\sin(2t)=0\end{aligned}$$

which gives $K_1 = 0$, $K_2 = -0.5$, $K_3 = 0.64$, and $K_4 = 0.48$.

(b) Homogeneous solution: The characteristic equation of the LTIC system (iii) is

$$s^2 + 2s + 1 = 0,$$

which has roots at $s = -1, -1$. The zero-input response is given by

$$y_{zi}(t) = Ae^{-t} + Bte^{-t}$$

for $t \geq 0$, with A and B being constants.

(c) Overall response of the system: The overall response of the system is obtained by summing up the above two responses, and use initial conditions to determine A and B, it is given by

$$y(t) = \left(3e^{-t} + 4te^{-t}\right)u(t) + \left(-0.64e^{-t} - 1.1te^{-t} - 0.5\sin(t) + 0.64\cos(2t) + 0.48\sin(2t)\right)u(t)$$

Problem 3.5

(ii) The output $y(t)$ is given by

$$y(t) = u(-t) * u(-t) = \int_{-\infty}^{\infty} u(-\tau)u(\tau - t)\, d\tau = \int_{-\infty}^{0} u(\tau - t)\, d\tau.$$

The output $y(t)$ is given by

$$y(t) = \int_{-\infty}^{0} u(\tau - t)\, d\tau = \begin{cases} 0 & \text{if}\,(t \geq 0) \\ \int_{t}^{0} u(\tau - t)\, d\tau & \text{if}\,(t < 0) \end{cases} = \begin{cases} 0 & \text{if}\,(t \geq 0) \\ -t & \text{if}\,(t < 0) \end{cases} = -tu(-t).$$

The aforementioned convolution can also be computed graphically.

(iv) The output $y(t)$ is given by

$$y(t) = e^{2t}u(-t) * e^{-3t}u(t) = \int_{-\infty}^{\infty} e^{2\tau}u(-\tau)e^{-3(t-\tau)}u(t-\tau)\, d\tau = e^{-3t}\int_{-\infty}^{0} e^{5\tau}u(t-\tau)\, d\tau.$$

Solving for the two cases ($t \geq 0$) and ($t < 0$), we get

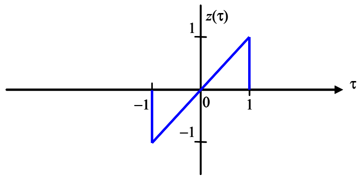
$$y(t) = e^{-3t}\int_{-\infty}^{0} e^{5\tau}u(t-\tau)\, d\tau = \begin{cases} e^{-3t}\int_{-\infty}^{t} e^{5\tau}\, d\tau & (t < 0) \\ e^{-3t}\int_{-\infty}^{0} e^{5\tau}\, d\tau & (t \geq 0) \end{cases} = \begin{cases} \frac{1}{5}e^{2t} & (t < 0) \\ \frac{1}{5}e^{-3t} & (t \geq 0). \end{cases}$$

Therefore, the output $y(t)$ is given by

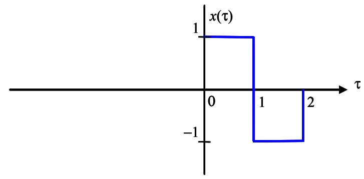
$$y(t) = \tfrac{1}{5}e^{2t}u(-t) + \tfrac{1}{5}e^{-3t}u(t).$$

Problem 3.6

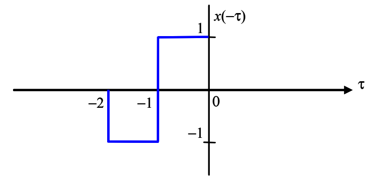
(ii) Using the graphical approach, the convolution of $x(t)$ with $z(t)$ is shown in Fig. S3.6.2, where we consider six different cases for different values of t.



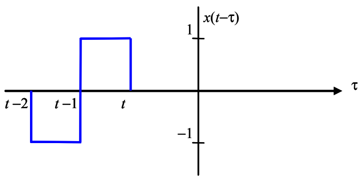
(a) Waveform for $z(\tau)$



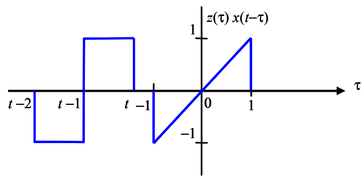
(b) Waveform for $x(\tau)$



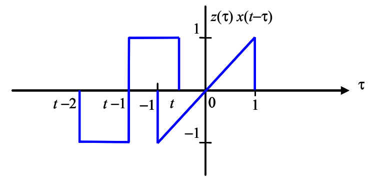
(c) Waveform for $x(-\tau)$



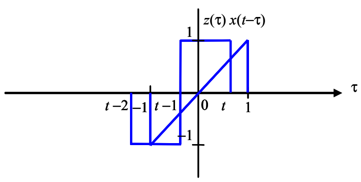
(d) Waveform for $x(t-\tau)$



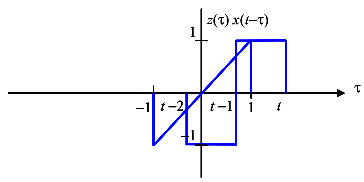
(e) Overlap btw $z(\tau)$ and $x(t-\tau)$ for ($t<-1$)



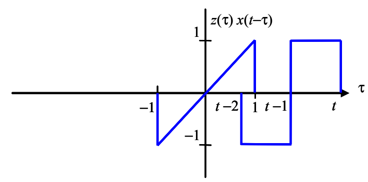
(f) Overlap btw $z(\tau)$ and $x(t-\tau)$ for ($-1\le t<0$)



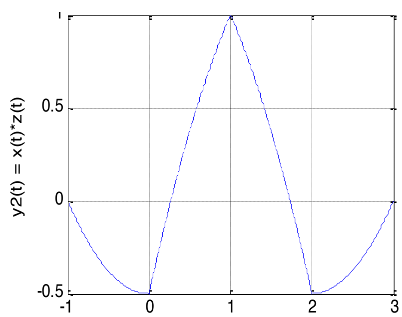
(g) Overlap btw $z(\tau)$ and $x(t-\tau)$ for ($0\le t<1$)



(h) Overlap btw $z(\tau)$ and $x(t-\tau)$ for ($1\le t<2$)



(i) Overlap btw $z(\tau)$ and $x(t-\tau)$ for ($2\le t<3$)



(j) Convolution output $y_2(t)$

Fig. S3.6.2: Convolution of $x(t)$ with $z(t)$ in Problem 3.6(ii).

Case I ($t<-1$): Since there is no overlap, $y_2(t)=0$.

Case II ($-1\le t<0$):
$$y_2(t)=\int_{-1}^{t}1.\tau d\tau=\tfrac{t^2}{2}-\tfrac{1}{2}.$$

Case III ($0\le t<1$):
$$y_2(t)=\int_{-1}^{t-1}(-1).\tau d\tau+\int_{t-1}^{t}1.\tau d\tau$$
$$=-\left(\tfrac{(t-1)^2}{2}-\tfrac{1}{2}\right)+\left(\tfrac{t^2}{2}-\tfrac{(t-1)^2}{2}\right)=-\tfrac{t^2}{2}+2t-\tfrac{1}{2}.$$

Case IV ($1 \le t < 2$):

$$y_2(t) = \int_{t-2}^{t-1} (-1).\tau\, d\tau + \int_{t-1}^{1} 1.\tau\, d\tau$$
$$= -\left(\tfrac{(t-1)^2}{2} - \tfrac{(t-2)^2}{2}\right) + \left(\tfrac{1}{2} - \tfrac{(t-1)^2}{2}\right) = -\tfrac{t^2}{2} + \tfrac{3}{2}.$$

Case V ($2 \le t < 3$):

$$y_2(t) = \int_{t-2}^{1} (-1).\tau d\tau = \frac{(t-2)^2}{2} - \frac{1}{2} = \frac{t^2}{2} - 2t + \frac{3}{2}.$$

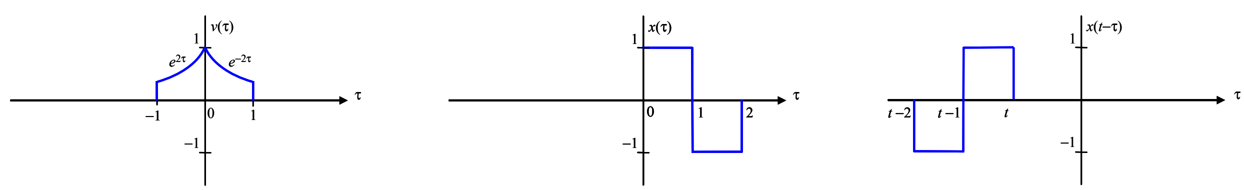
Case VI ($t > 4$): Since there is no overlap, $y_2(t) = 0$.

Combining all the cases, the result of the convolution $y_2(t) = x(t) * z(t)$ is given by

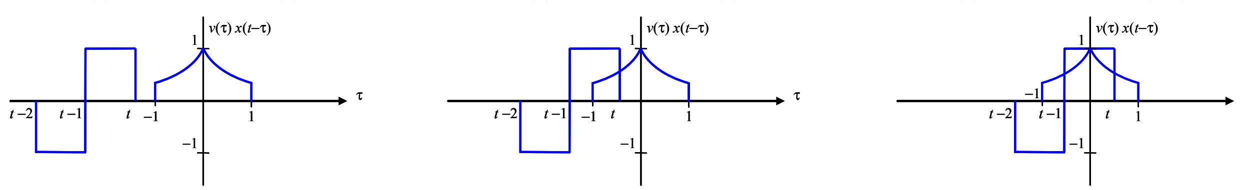
$$y_2(t) = \begin{cases} \frac{t^2}{2} - \frac{1}{2} & (-1 \le t < 0) \\ -\frac{t^2}{2} + 2t - \frac{1}{2} & (0 \le t < 1) \\ -\frac{t^2}{2} + \frac{3}{2} & (1 \le t < 2) \\ \frac{t^2}{2} - 2t + \frac{3}{2} & (2 \le t < 3) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_2(t)$ plotted in Fig. S3.6.2(j).

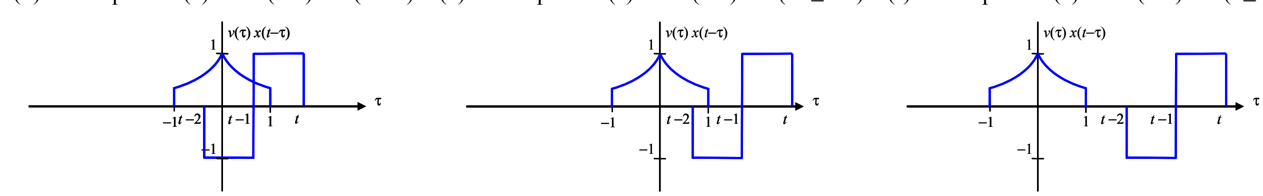
(iv) Using the graphical approach, the convolution of $x(t)$ with $v(t)$ is shown in Fig. 3.6.4, where we consider six different cases for different values of t.



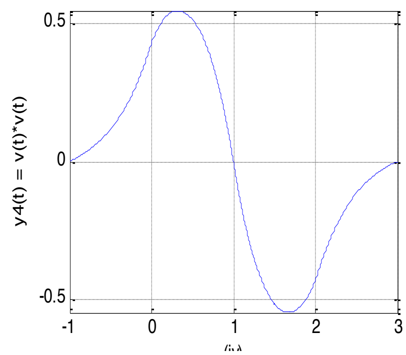
(a) Waveform for $v(\tau)$ (b) Waveform for $x(\tau)$ (c) Waveform for $x(t-\tau)$



(d) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($t<-1$) (e) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($-1\le t<0$) (f) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($0\le t<1$)



(g) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($1\le t<2$) (h) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($2\le t<3$) (i) Overlap btw $v(\tau)$ and $x(t-\tau)$ for ($t>3$)



(j) Convolution output $y_4(t)$

Fig. S3.6.4: Convolution of $x(t)$ with $v(t)$ in Problem 3.6(iv).

Case I ($t < -1$): Since there is no overlap, $y_4(t) = 0$.

Case II ($-1 \le t < 0$): $$y_4(t) = \int_{-1}^{t} 1.e^{2\tau} d\tau = \tfrac{1}{2}\left(e^{2t} - e^{-2}\right).$$

Case III ($0 \le t < 1$): $$\begin{cases} y_4(t) = \int_{-1}^{t-1} (-1).e^{2\tau} d\tau + \int_{t-1}^{0} 1.e^{2\tau} d\tau + \int_{0}^{t} 1.e^{-2\tau} d\tau \\ \quad = -\tfrac{1}{2}\left(e^{2(t-1)} - e^{-2}\right) + \tfrac{1}{2}\left(1 - e^{2(t-1)}\right) + \tfrac{1}{2}\left(1 - e^{-2t}\right) \\ \quad = -e^{2(t-1)} + \tfrac{1}{2}e^{-2} + 1 - \tfrac{1}{2}e^{-2t}. \end{cases}$$

Case IV ($1 \le t < 2$): $$\begin{cases} y_4(t) = \int_{t-2}^{0} (-1).e^{2\tau} d\tau + \int_{0}^{t-1} (-1).e^{-2\tau} d\tau + \int_{t-1}^{1} 1.e^{-2\tau} d\tau \\ \quad = -\tfrac{1}{2}\left(1 - e^{2(t-2)}\right) + \tfrac{1}{2}\left(e^{-2(t-1)} - 1\right) + \tfrac{1}{(-2)}\left(e^{-2} - e^{-2(t-1)}\right) \\ \quad = \tfrac{1}{2}e^{2(t-2)} - 1 - \tfrac{1}{2}e^{-2} + e^{-2(t-1)}. \end{cases}$$

Case V ($2 \le t < 3$): $$y_4(t) = \int_{t-2}^{1} (-1).e^{-2\tau} d\tau = \tfrac{1}{2}\left(e^{-2} - e^{-2(t-2)}\right).$$

Case VI ($t > 4$): Since there is no overlap, $y_4(t) = 0$.

Combining all the cases, the result of the convolution $y_4(t) = x(t) * v(t)$ is given by

$$y_4(t)=\begin{cases} \frac{1}{2}e^{2t}-\frac{1}{2}e^{-2} & (-1\le t<0) \\ -e^{2(t-1)}+\frac{1}{2}e^{-2}+1-\frac{1}{2}e^{-2t} & (0\le t<1) \\ \frac{1}{2}e^{2(t-2)}-1-\frac{1}{2}e^{-2}+e^{-2(t-1)} & (1\le t<2) \\ \frac{1}{2}e^{-2}-\frac{1}{2}e^{-2(t-2)} & (2\le t<3) \\ 0 & \text{elsewhere.} \end{cases}$$

The output is $y_4(t)$ plotted in Fig. S3.6.4(j).

Problem 3.12

(i) System $h1(t)$ is NOT memoryless since $h1(t)\neq 0$ for $t\neq 0$.
System $h1(t)$ is causal since $h1(t)=0$ for $t<0$.
System $h1(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty}|h1(t)|\,dt=\int_{-\infty}^{\infty}\delta(t)dt+\int_{-\infty}^{\infty}e^{-5t}u(t)dt=1+\left[-\tfrac{1}{5}e^{-5t}\right]_0^{\infty}=\frac{6}{5}<\infty.$$

(ii) System $h2(t)$ is NOT memoryless since $h3(t)\neq 0$ for $t\neq 0$.
System $h2(t)$ is causal since $h2(t)=0$ for $t<0$.
System $h2(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty}|h2(t)|\,dt=\int_{-\infty}^{\infty}e^{-2t}u(t)dt=\int_{0}^{\infty}e^{-2t}dt=\left[-\tfrac{1}{2}e^{-2t}\right]_0^{\infty}=\frac{1}{2}<\infty.$$

(iii) System $h3(t)$ is NOT memoryless since $h3(t)\neq 0$ for $t\neq 0$.
System $h3(t)$ is causal since $h3(t)=0$ for $t<0$.
System $h3(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty}|h3(t)|\,dt=\int_{-\infty}^{\infty}e^{-5t}\sin(2\pi t)u(t)dt=\int_{0}^{\infty}e^{-5t}\sin(2\pi t)dt<\infty.$$

(iv) System $h4(t)$ is NOT memoryless since $h4(t)\neq 0$ for $t\neq 0$.
System $h4(t)$ is NOT causal since $h4(t)\neq 0$ for $t<0$.
System $h4(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty}|h4(t)|\,dt=\int_{-\infty}^{0}e^{2t}dt+\int_{0}^{\infty}e^{-2t}dt+\int_{-1}^{1}1\,dt=3<\infty.$$

(v) System $h5(t)$ is NOT memoryless since $h5(t)\neq 0$ for $t\neq 0$.
System $h5(t)$ is NOT causal since $h5(t)\neq 0$ for $t<0$.
System $h5(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty}|h5(t)|\,dt=\int_{-4}^{4}t\,dt=\left.\frac{t^2}{2}\right|_{-4}^{4}=16<\infty.$$

(vi) System $h6(t)$ is NOT memoryless since $h6(t)\neq 0$ for $t\neq 0$.
System $h6(t)$ is NOT causal since $h6(t)\neq 0$ for $t<0$.
System $h6(t)$ is NOT BIBO stable since

$$\int_{-\infty}^{\infty} | h6(t) | \, dt = \int_{-\infty}^{\infty} | \sin(10t) | \, dt = \infty .$$

Consider the bounded input signal $\sin(10t)$. If this signal is applied to the system, the output can be calculated as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} \sin(10\tau)\sin(10t - 10\tau) d\tau$$

The output at t=0 is given by,

$$y(0) = \int_{-\infty}^{\infty} \sin(10\tau)\sin(-10\tau)\, d\tau = -\int_{-\infty}^{\infty} \sin^2(10\tau)\, d\tau = -\tfrac{1}{2}\int_{-\infty}^{\infty} \big(1 - \cos(20\tau)\big) d\tau$$

$$= -\tfrac{1}{2}\underbrace{\int_{-\infty}^{\infty} d\tau}_{=\infty} + \tfrac{1}{2}\underbrace{\int_{-\infty}^{\infty} \cos(20\tau)\, d\tau}_{=\textit{finite value}} = -\infty$$

It is observed that the output becomes unbounded even if the input is always bounded. This is because the system is not BIBO stable.

(vii) System $h7(t)$ is NOT memoryless since $h7(t) \neq 0$ for $t \neq 0$.
System $h7(t)$ is causal since $h7(t) = 0$ for $t < 0$.
System $h7(t)$ is NOT BIBO stable since

$$\int_{-\infty}^{\infty} | h7(t) | \, dt = \int_{0}^{\infty} \cos(5t) dt = \infty .$$

Consider the bounded input signal $\cos(5t)$. If this signal is applied to the system, the output can be calculated as:

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau = \int_{-\infty}^{\infty} \cos(5t - 5\tau)\cos(5\tau) u(\tau) d\tau = \int_{0}^{\infty} \cos(5t - 5\tau)\cos(5\tau) d\tau \cdot$$

The output at t=0 is given by,

$$y(0) = \int_{0}^{\infty} \cos(-5\tau)\cos(5\tau)\, d\tau = \int_{0}^{\infty} \cos^2(5\tau)\, d\tau = \tfrac{1}{2}\int_{0}^{\infty} \big(1 + \cos(10\tau)\big) d\tau$$

$$= \tfrac{1}{2}\underbrace{\int_{0}^{\infty} d\tau}_{=\infty} + \tfrac{1}{2}\underbrace{\int_{0}^{\infty} \cos(10\tau)\, d\tau}_{=\textit{finite value}} = \infty$$

It is observed that the output becomes unbounded at t=0 even if the input is always bounded. This proves that the system is not BIBO stable.

(viii) System $h8(t)$ is NOT memoryless since $h8(t) \neq 0$ for $t \neq 0$.
System $h8(t)$ is NOT causal since $h8(t) \neq 0$ for $t < 0$.
System $h8(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h8(t)|\,dt = \int_{-\infty}^{\infty} 0.95^{|t|}\,dt = 2\int_{0}^{\infty} 0.95^{t}\,dt = 2\int_{0}^{\infty} e^{t\ln(0.95)}\,dt = \frac{2}{\ln(0.95)}\left[e^{t\ln(0.95)}\right]_0^{\infty}$$

$$= \frac{2}{\ln(0.95)}\left[0-1\right] = -\frac{2}{\ln(0.95)} = 39 < \infty$$

(ix) System $h9(t)$ is NOT memoryless since $h8(t) \neq 0$ for $t \neq 0$.
System $h9(t)$ is NOT causal since $h8(t) = 0$ for $t < 0$.
System $h8(t)$ is BIBO stable since

$$\int_{-\infty}^{\infty} |h9(t)|\,dt = \int_{-1}^{1} 1dt = 2 < \infty.$$

Problem 3.14

(i) System (i) is invertible with the impulse response $h1_i(t)$ of its inverse system given by

$$h1_i(t) = \tfrac{1}{5}\delta(t+2).$$

(ii) System (ii) will be invertible if there exists an impulse response $h2_i(t)$ such that

$$h2(t) * h2_i(t) = \delta(t).$$

Substituting the value of $h2(t)$, we get

$$h2_i(t) + h2_i(t+2) = \delta(t)$$

which simplifies to $$h2_i(t) = \delta(t-2) - h2_i(t-2).$$

Substituting the value of $h2_i(t-2) = \delta(t-4) - h2_i(t-4)$ in the earlier expression gives

$$h2_i(t) = \delta(t-2) - \delta(t-4) + h2_i(t-4).$$

Iterating the above procedure yields,

$$h2_i(t) = \sum_{m=1}^{\infty} (-1)^{m+1}\delta(t-2m).$$

Therefore, the system is invertible with the impulse response of the inverse system given above.

(iii) System (iii) will be invertible if there exists an impulse response $h3_i(t)$ such that

$$h3(t) * h3_i(t) = \delta(t).$$

Substituting the value of $h3(t)$, we get

$$h3_i(t+1) + h3_i(t-1) = \delta(t)$$

which simplifies to $$h3_i(t) = \delta(t-1) - h3_i(t-2).$$

Substituting the value of $h3_i(t-2) = \delta(t-3) - h3_i(t-4)$ in the earlier expression yields

$$h3_i(t) = \delta(t-1) - \delta(t-3) + h3_i(t-4).$$

Iterating the above procedure yields,

$$h3_i(t) = \sum_{m=1}^{\infty} (-1)^{m+1} \delta(t+1-2m).$$

(iv) System (iv) will be invertible if there exists an impulse response $h4_i(t)$ such that

$$h4(t) * h4_i(t) = \delta(t).$$

Substituting the value of $h4(t)$, we get

$$\int_{-\infty}^{\infty} h4_i(\tau) u(t-\tau) d\tau = \delta(t)$$

which simplifies to $$\int_{-\infty}^{t} h4_i(\tau) d\tau = \delta(t).$$

Differentiating both sides of the above expression with respect to t, we obtain

$$h4_i(t) = \tfrac{d}{dt}\left(\delta(t)\right).$$

In other words, system (iv) is an integrator. As expected, its inverse system is a differentiator.

(v) System (v) will be invertible if there exists an impulse response $h5_i(t)$ such that

$$h5(t) * h5_i(t) = \delta(t).$$

Substituting the value of $h5(t)$, we obtain

$$\int_{-\infty}^{\infty} h5_i(\tau) \mathrm{rect}(\tfrac{t-\tau}{4}) d\tau = \delta(t),$$

which simplifies to $$\int_{t-4}^{t+4} h5_i(\tau) d\tau = \delta(t),$$

which is expressed as $$\underbrace{\int_{-\infty}^{t+4} h5_i(\tau) d\tau}_{\text{Substitute } \alpha=\tau-4} - \underbrace{\int_{-\infty}^{t-4} h5_i(\tau) d\tau}_{\text{Substitute } \alpha=\tau+4} = \delta(t),$$

or, $$\int_{-\infty}^{t} h5_i(\alpha+4) d\alpha - \int_{-\infty}^{t} h5_i(\alpha-4) d\alpha = \delta(t)$$

Taking the derivative of both sides of the equation with respect to t, we obtain

$$h5_i(t+4) - h5_i(t-4) = \tfrac{d}{dt}\left(\delta(t)\right).$$

which can be expressed as

$$h5_i(t) = \sum_{m=0}^{\infty} \tfrac{d}{dt}\left(\delta(t-4-8m)\right).$$

(vi) System (vi) will be invertible if there exists an impulse response $h6_i(t)$ such that

$$h6(t) * h6_i(t) = \delta(t).$$

Substituting the value of $h6(t)$, we obtain

$$\int_{-\infty}^{\infty} h6_i(\tau) e^{-2(t-\tau)} u(t-\tau) d\tau = \delta(t),$$

which simplifies to

$$e^{-2t} \int_{-\infty}^{t} h6_i(\tau) e^{2\tau} d\tau = \delta(t)$$

or,

$$\int_{-\infty}^{t} h6_i(\tau) e^{2\tau} d\tau = \delta(t) e^{2t}.$$

Taking the derivative of both sides of the equation with respect to t, we obtain

$$h6_i(t) e^{2t} = \tfrac{d}{dt}\left(\delta(t) e^{2t}\right) = e^{2t} \tfrac{d}{dt}\left(\delta(t)\right) + 2\delta(t) e^{2t}$$

or,

$$h6_i(t) = \tfrac{d}{dt}\left(\delta(t)\right) + 2\delta(t).$$

$$b_n = \frac{2}{T}\int_0^T \left(1 - \frac{t}{T}\right) \sin(n\omega_0 t) dt$$

$$= \frac{2}{T}\left[\left(1 - \frac{t}{T}\right) \times \frac{-\cos(n\omega_0 t)}{(n\omega_0)} - \left(-\frac{1}{T}\right) \times \frac{-\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^T$$

$$= \frac{2}{T}\left[0 - (1) \times \frac{-1}{(n\omega_0)} - \left(\frac{1}{T}\right) \times \frac{\sin(n\omega_0 T)}{(n\omega_0)^2} + -\left(\frac{1}{T}\right) \times \frac{\sin(0)}{(n\omega_0)^2}\right]$$

$$= \frac{2}{n\omega_0 T} = \frac{1}{n\pi}$$

(e) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.30), the CTFS coefficient T_0 is given by

$$a_0 = \tfrac{1}{2T}\int_0^{2T} x(t) dt = \tfrac{1}{2T}\int_0^{T}\left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right] dt = \tfrac{1}{2T}\int_0^T dt - \tfrac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right) dt$$

$$= \tfrac{1}{2} + \tfrac{1}{4T} \times \tfrac{1}{\pi/T}\left[\cos(\tfrac{\pi t}{T})\right]_0^T = \tfrac{1}{2} + \tfrac{1}{4\pi}\left[\cos(\pi) - \cos(0)\right] = \tfrac{1}{2} - \tfrac{1}{2\pi} = \tfrac{\pi - 1}{2\pi}$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for ($n \neq 0$), are given by

$$a_n = \tfrac{2}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right] \cos(n\omega_0 t) dt = \underbrace{\tfrac{1}{T}\int_0^T \cos(n\omega_0 t) dt}_{=A} - \underbrace{\tfrac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t) dt}_{=B}$$

where Integrals A and B are simplified as

$$A = \tfrac{1}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^T = \tfrac{1}{n\pi}\left[\sin(n\omega_0 T) - 0\right] = \tfrac{1}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

and

$$B = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t)dt = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos\left(\tfrac{n\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[\sin\tfrac{\pi t}{T}(n+1) - \sin\left(\tfrac{\pi t}{T}(n-1)\right)\right]dt$$

$$= \frac{1}{4T}\times\frac{-1}{\pi(n+1)/T}\left[\cos\tfrac{\pi t}{T}(n+1)\right]_0^T + \frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\cos\tfrac{\pi t}{T}(n-1)\right]_0^T \qquad [\text{for } n\neq 1]$$

$$= \frac{1}{4\pi(n+1)}\left[1-\cos\pi(n+1)\right] - \frac{1}{4\pi(n-1)}\left[1-\cos\pi(n-1)\right]$$

$$= \begin{cases} 0 & 1\neq n = odd \\ \frac{2}{4\pi(n+1)} - \frac{2}{4\pi(n-1)} & n = even \end{cases} = \begin{cases} 0 & 1\neq n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

For $n=1,\ B = \frac{1}{4T}\int_0^T \sin\frac{2\pi t}{T}dt = \frac{1}{4T}\times\frac{-1}{2\pi/T}\left[\cos\frac{2\pi t}{T}\right]_0^T = \frac{1}{8\pi}\left[1-\cos 2\pi\right] = 0.$

In other words,

$$B = \begin{cases} 0 & n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

which implies that

$$a_n = A - B = \begin{cases} 0 & n = odd \\ \frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{2T}\int_0^T \left[1-0.5\sin\left(\tfrac{\pi t}{T}\right)\right]\sin(n\omega_0 t)dt = \underbrace{\frac{1}{T}\int_0^T \sin(n\omega_0 t)dt}_{=C} - \underbrace{\frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\sin(n\omega_0 t)dt}_{=D}$$

where Integrals C and D are simplified as

$$C = \frac{1}{n\omega_0 T}\left[-\cos(n\omega_0 t)\right]_0^T = \frac{1}{n\pi}\left[-\cos(n\omega_0 T)+\cos(0)\right] = \frac{1}{n\pi}\left[1-\cos(n\pi)\right] = \begin{cases} 0 & n = even \\ \frac{2}{n\pi} & n = odd \end{cases}$$

and

$$D = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\sin\left(\tfrac{n\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[\cos\tfrac{\pi t}{T}(n-1) - \cos\left(\tfrac{\pi t}{T}(n+1)\right)\right]dt$$

$$= \frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\sin\tfrac{\pi t}{T}(n-1)\right]_0^T - \frac{1}{4T}\times\frac{1}{\pi(n+1)/T}\left[\sin\tfrac{\pi t}{T}(n+1)\right]_0^T \qquad [\text{for } n\neq 1]$$

$$= \frac{1}{4\pi(n-1)}\left[\sin\pi(n-1) - \sin(0)\right] - \frac{1}{4\pi(n+1)}\left[\sin\pi(n+1) - \sin(0)\right]$$

$$= 0 \qquad [\text{for } n\neq 1]$$

For (n = 1),

$$D = \frac{1}{2T}\int_0^T \sin^2\left(\tfrac{\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[1-\cos\left(\tfrac{2\pi t}{T}\right)\right]dt = \left(\frac{1}{4} - \frac{1}{4T\times 2\pi/T}\underbrace{\left[\sin\tfrac{2\pi t}{T}\right]_0^T}_{=0}\right) = \frac{1}{4}.$$

In other words,

$$D = \begin{cases} \frac{1}{4} & n = 1 \\ 0 & n > 1 \end{cases}.$$

Therefore,

$$b_n = C - D = \begin{cases} 0 & n = even \\ \frac{2}{\pi} - \frac{1}{4} & n = 1 \\ \frac{2}{n\pi} & 1 \neq n = odd. \end{cases}$$ ∎

$$D_n = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \pm j\left(\frac{1}{8} - \frac{1}{\pi}\right) & n = \pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \frac{1}{jn\pi} & \pm 1 \neq n = odd. \end{cases}$$