Instructors Solutions to Assignment 3

Problem 4.6

(a) By inspection, we note that the time period $T_0 = 2\pi$, which implies that the fundamental frequency $\omega_0 = 1$.

Since the CTFS coefficient a_0 represents the average value of the signal, therefore, $a_0 = 3/2$.

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for ($n \neq 0$), are given by

$$a_n = \frac{2}{T_0}\int_0^{T_0} x1(t)\cos(n\omega_0 t)dt = \frac{1}{\pi}\int_0^{\pi} 3\cos(n\omega_0 t)dt = \frac{1}{\pi}\int_0^{\pi} 3\cos(nt)dt$$

$$= \frac{3}{n\pi}\left[\sin(nt)\right]_0^{\pi} \quad = \frac{3}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{T_0}\int_0^{T_0} x1(t)\sin(n\omega_0 t)dt = \frac{1}{\pi}\int_0^{\pi} 3\sin(nt)dt = \frac{3}{n\pi}\left[-\cos(nt)\right]_0^{\pi} \quad = \frac{3}{n\pi}\left[-\cos(n\pi) + \cos(0)\right] = \frac{3}{n\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} \frac{6}{n\pi} & n = odd \\ 0 & n = even \end{cases}$$

(c) By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = 2\pi/T$.

Since the CTFS coefficient a_0 represents the average value of the signal, therefore, $a_0 = 1/2$.

Since the function $[x3(t) - 0.5]$ is odd, therefore, the CTFS cosine coefficients $a_n = 0$, for ($n \neq 0$).

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{T}\int_0^{T}\left(1 - \frac{t}{T}\right)\sin(n\omega_0 t)dt$$

$$= \frac{2}{T}\left[\left(1 - \frac{t}{T}\right)\times\frac{-\cos(n\omega_0 t)}{(n\omega_0)} - \left(-\frac{1}{T}\right)\times\frac{-\sin(n\omega_0 t)}{(n\omega_0)^2}\right]_0^T$$

$$= \frac{2}{T}\left[0 - (1)\times\frac{-1}{(n\omega_0)} - \left(\frac{1}{T}\right)\times\frac{\sin(n\omega_0 T)}{(n\omega_0)^2} + -\left(\frac{1}{T}\right)\times\frac{\sin(0)}{(n\omega_0)^2}\right]$$

$$= \frac{2}{n\omega_0 T} = \frac{1}{n\pi}$$

(e) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.30), the CTFS coefficient T_0 is given by

$$a_0 = \frac{1}{2T}\int_0^{2T} x(t)dt = \frac{1}{2T}\int_0^{T}\left[1 - 0.5\sin\left(\frac{\pi t}{T}\right)\right]dt = \frac{1}{2T}\int_0^{T} dt - \frac{1}{4T}\int_0^{T}\sin\left(\frac{\pi t}{T}\right)dt$$

$$= \frac{1}{2} + \frac{1}{4T}\times\frac{1}{\pi/T}\left[\cos\left(\frac{\pi t}{T}\right)\right]_0^T = \frac{1}{2} + \frac{1}{4\pi}\left[\cos(\pi) - \cos(0)\right] = \frac{1}{2} - \frac{1}{2\pi} = \frac{\pi - 1}{2\pi}$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for ($n \neq 0$), are given by

$$a_n = \frac{2}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]\cos(n\omega_0 t)dt = \underbrace{\frac{1}{T}\int_0^T \cos(n\omega_0 t)dt}_{=A} - \underbrace{\frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t)dt}_{=B}$$

where Integrals A and B are simplified as

$$A = \frac{1}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^T = \frac{1}{n\pi}\left[\sin(n\omega_0 T) - 0\right] = \frac{1}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

and

$$B = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos(n\omega_0 t)dt = \frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\cos\left(\tfrac{n\pi t}{T}\right)dt = \frac{1}{4T}\int_0^T \left[\sin\tfrac{\pi t}{T}(n+1) - \sin\left(\tfrac{\pi t}{T}(n-1)\right)\right]dt$$

$$= \frac{1}{4T}\times\frac{-1}{\pi(n+1)/T}\left[\cos\tfrac{\pi t}{T}(n+1)\right]_0^T + \frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\cos\tfrac{\pi t}{T}(n-1)\right]_0^T \qquad [\text{for } n \neq 1]$$

$$= \frac{1}{4\pi(n+1)}\left[1 - \cos\pi(n+1)\right] - \frac{1}{4\pi(n-1)}\left[1 - \cos\pi(n-1)\right]$$

$$= \begin{cases} 0 & 1 \neq n = odd \\ \frac{2}{4\pi(n+1)} - \frac{2}{4\pi(n-1)} & n = even \end{cases} = \begin{cases} 0 & 1 \neq n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

For $\quad n = 1,\ B = \frac{1}{4T}\int_0^T \sin\frac{2\pi t}{T}\,dt = \frac{1}{4T}\times\frac{-1}{2\pi/T}\left[\cos\frac{2\pi t}{T}\right]_0^T = \frac{1}{8\pi}\left[1 - \cos 2\pi\right] = 0\cdot$

In other words, $\quad B = \begin{cases} 0 & n = odd \\ -\frac{1}{\pi(n^2-1)} & n = even \end{cases}$

which implies that

$$a_n = A - B = \begin{cases} 0 & n = odd \\ \frac{1}{\pi(n^2-1)} & n = even \end{cases}$$

Using Eq. (4.32), the CTFS sine coefficients b_n's are given by

$$b_n = \frac{2}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]\sin(n\omega_0 t)\,dt = \underbrace{\frac{1}{T}\int_0^T \sin(n\omega_0 t)\,dt}_{=C} - \underbrace{\frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\sin(n\omega_0 t)\,dt}_{=D}$$

where Integrals C and D are simplified as

$$C = \frac{1}{n\omega_0 T}\left[-\cos(n\omega_0 t)\right]_0^T = \frac{1}{n\pi}\left[-\cos(n\omega_0 T) + \cos(0)\right] = \frac{1}{n\pi}\left[1 - \cos(n\pi)\right] = \begin{cases} 0 & n = even \\ \frac{2}{n\pi} & n = odd \end{cases}$$

and

$$D=\frac{1}{2T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)\sin\left(\tfrac{n\pi t}{T}\right)dt=\frac{1}{4T}\int_0^T\left[\cos\tfrac{\pi t}{T}(n-1)-\cos\left(\tfrac{\pi t}{T}(n+1)\right)\right]dt$$

$$=\frac{1}{4T}\times\frac{1}{\pi(n-1)/T}\left[\sin\tfrac{\pi t}{T}(n-1)\right]_0^T-\frac{1}{4T}\times\frac{1}{\pi(n+1)/T}\left[\sin\tfrac{\pi t}{T}(n+1)\right]_0^T \qquad [\text{for } n\neq 1]$$

$$=\frac{1}{4\pi(n-1)}\left[\sin\pi(n-1)-\sin(0)\right]-\frac{1}{4\pi(n+1)}\left[\sin\pi(n+1)-\sin(0)\right]$$

$$=0 \qquad [\text{for } n\neq 1]$$

For ($n = 1$),

$$D=\frac{1}{2T}\int_0^T \sin^2\left(\tfrac{\pi t}{T}\right)dt=\frac{1}{4T}\int_0^T\left[1-\cos\left(\tfrac{2\pi t}{T}\right)\right]dt=\left(\frac{1}{4}-\frac{1}{4T\times 2\pi/T}\underbrace{\left[\sin\tfrac{2\pi t}{T}\right]_0^T}_{=0}\right)=\frac{1}{4}\,.$$

In other words,

$$D=\begin{cases}\frac{1}{4} & n=1\\ 0 & n>1\end{cases}.$$

Therefore,

$$b_n=C-D=\begin{cases}0 & n=even\\ \frac{2}{\pi}-\frac{1}{4} & n=1\\ \frac{2}{n\pi} & 1\neq n=odd.\end{cases}$$

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Problem 4.11

(a) By inspection, we note that the time period $T_0 = 2\pi$, which implies that the fundamental frequency $\omega_0 = 1$. Using Eq. (4.44), the DTFS coefficients D_n's are given by

$$D_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)e^{-jn\omega_0 t}dt=\frac{1}{2\pi}\int_0^{\pi}3e^{-jnt}dt=\begin{cases}\frac{3}{2}, & n=0\\ \frac{3}{j2n\pi}\left(1-e^{-jn\pi}\right) & n\neq 0.\end{cases}$$

or,

$$D_n=\frac{3}{j2n\pi}\left(1-(-1)^n\right)=\begin{cases}\frac{3}{2} & n=0\\ 0 & \text{even } n, n\neq 0.\\ \frac{3}{jn\pi} & \text{odd } n\end{cases}$$

The magnitude and phase spectra are given by

Magnitude Spectrum:

$$|D_n|=\begin{cases}\frac{3}{2}, & n=0\\ 0, & \text{even } n, n\neq 0.\\ \frac{3}{n\pi}, & \text{odd } n.\end{cases}$$

Phase Spectrum:

$$\angle D_n=\begin{cases}0, & \text{even } n\\ -\frac{\pi}{2}, & \text{odd } n, n>0\\ \frac{\pi}{2}, & \text{odd } n, n<0.\end{cases}$$

The magnitude and phase spectra are shown in row 1 of the subplots included in Fig. S4.11.

(c) By inspection, we note that the time period $T_0 = T$, which implies that the fundamental frequency $\omega_0 = 2\pi/T$. Using Eq. (4.44), the DTFS coefficients D_n's are given by

$$D_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt = \frac{1}{T}\int_0^T \left(1-\tfrac{t}{T}\right)e^{-jn\omega_0 t}dt = \begin{cases} \tfrac{1}{T}\times\tfrac{T}{2} = \tfrac{1}{2}, & n = 0 \\ \dfrac{1}{T}\displaystyle\int_0^T \left(1-\tfrac{t}{T}\right)e^{-jn\omega_0 t}dt & n \neq 0. \end{cases}$$

For ($n \neq 0$), the DTFS coefficients are given by

$$D_n = \frac{1}{T}\int_0^T \left(1-\tfrac{t}{T}\right)e^{-jn\omega_0 t}dt = \left[\left(1-\tfrac{t}{T}\right)\frac{e^{-jn\omega_0 t}}{(-jn\omega_0)} - \left(-\tfrac{1}{T}\right)\frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2}\right]_0^T,$$

which reduces to

$$D_n = \left[0 - \tfrac{1}{T}\frac{1}{(-jn\omega_0)} + \tfrac{1}{T}\frac{e^{-jn\omega_0 T}}{(-jn\omega_0)^2} - \tfrac{1}{T}\frac{1}{(-jn\omega_0)^2}\right]_0^T = \frac{1}{j2n\pi}.$$

Combining the two cases, we get

$$D_n = \begin{cases} \tfrac{1}{2}, & n = 0 \\ \tfrac{1}{j2n\pi}, & n \neq 0. \end{cases}$$

The magnitude and phase spectra are given by

Magnitude Spectrum: $$|D_n| = \begin{cases} \tfrac{1}{2}, & n = 0 \\ \tfrac{1}{2|n|\pi}, & n \neq 0. \end{cases}$$

Phase Spectrum: $$\prec D_n = \begin{cases} 0, & n = 0 \\ 0.5\pi, & n < 0 \\ -0.5\pi, & n > 0. \end{cases}$$

The magnitude and phase spectra are shown in row 3 of the subplots included in Fig. S4.11.

(e) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$. For ($n = 0$), the exponential DTFS coefficients is given by

$$D_0 = \tfrac{1}{2T}\int_0^{2T} x(t)dt = \tfrac{1}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]dt = \tfrac{1}{2T}\int_0^T dt - \tfrac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)dt$$
$$= \tfrac{1}{2} + \tfrac{1}{4T}\times\tfrac{1}{\pi/T}\left[\cos(\tfrac{\pi t}{T})\right]_0^T = \tfrac{1}{2} + \tfrac{1}{4\pi}\left[\cos(\pi) - \cos(0)\right] = \tfrac{1}{2} - \tfrac{1}{2\pi}$$

For ($n = 0$), the exponential DTFS coefficients is given by

$$D_n = \tfrac{1}{2T}\int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]e^{-jn\omega_0 t}dt = \underbrace{\tfrac{1}{2T}\int_0^T e^{-jn\omega_0 t}dt}_{=A} - \underbrace{\tfrac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)e^{-jn\omega_0 t}dt}_{=B}\cdot$$

Solving for Integrals A and B, we get

$$A=\frac{1}{2T}\int_0^T e^{-jn\omega_0 t}dt=\frac{1}{-j2n\omega_0 T}\left[e^{-jn\omega_0 t}\right]_0^T=\frac{1}{-j2n\pi}\left[e^{-jn\pi}-1\right]=\frac{1}{j2n\pi}\left[1-(-1)^n\right]$$

and

$$B=\frac{1}{4T}\int_0^T \sin\left(\tfrac{\pi t}{T}\right)e^{\frac{-jn\pi t}{T}}dt=\frac{1}{j8T}\int_0^T\left[e^{\frac{j\pi t}{T}}-e^{\frac{-j\pi t}{T}}\right]e^{\frac{-jn\pi t}{T}}dt=\frac{1}{j8T}\int_0^T\left[e^{\frac{-j(n-1)\pi t}{T}}-e^{\frac{-j(n+1)\pi t}{T}}\right]dt$$

$$=\frac{1}{j8T}\left[\tfrac{T}{-j(n-1)\pi}e^{\frac{-j(n-1)\pi t}{T}}+\tfrac{T}{j(n+1)\pi}e^{\frac{-j(n+1)\pi t}{T}}\right]_0^T \qquad \text{for } n\neq\pm1$$

$$=\frac{1}{8}\left[\tfrac{1}{(n-1)\pi}\left(e^{-j(n-1)\pi}-1\right)-\tfrac{1}{(n+1)\pi}\left(e^{-j(n+1)\pi}-1\right)\right]$$

$$=\frac{1}{8\pi}\left[\tfrac{1}{(n-1)}-\tfrac{1}{(n+1)}\right]\left[(-1)^{(n-1)}-1\right]=\tfrac{-1}{4\pi(n^2-1)}\left[1-(-1)^{(n-1)}\right]$$

$$=\tfrac{-1}{4\pi(n^2-1)}\left[1+(-1)^n\right]$$

For $n=\pm 1$, Integral B reduces to

$$\text{For } n=1,\quad B=\frac{1}{j8T}\int_0^T\left[1-e^{\frac{-j2\pi t}{T}}\right]dt=\frac{1}{j8T}\left[t+\tfrac{T}{j2\pi}e^{\frac{-j2\pi t}{T}}\right]_0^T=\frac{T}{j8T}=\frac{1}{j8}$$

and

$$\text{For } n=-1,\quad B=\frac{1}{j8T}\int_0^T\left[e^{\frac{-j2\pi t}{T}}-1\right]dt=\frac{1}{j8T}\left[-\tfrac{T}{j2\pi}e^{\frac{-j2\pi t}{T}}-t\right]_0^T=\frac{-T}{j8T}=-\frac{1}{j8}.$$

In other words,

$$B=\begin{cases}\pm\frac{1}{j8} & n=\pm1\\ \frac{-1}{4\pi(n^2-1)}\left[1+(-1)^n\right] & \text{otherwise}\end{cases}$$

Combining, the above cases, the CTFS coefficients can be expressed as

$$D_n=\begin{cases}\frac{1}{2}-\frac{1}{2\pi} & n=0\\ \frac{1}{j2n\pi}\left[1-(-1)^n\right]\mp\frac{1}{j8} & n=\pm1\\ \frac{1}{j2n\pi}\left[1-(-1)^n\right]+\frac{1}{4\pi(n^2-1)}\left[1+(-1)^n\right] & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\left(1-\frac{1}{\pi}\right) & n=0\\ \mp j\left(\frac{1}{\pi}-\frac{1}{8}\right) & n=\pm1\\ \frac{1}{4\pi(n^2-1)}\left[1+(-1)^n\right]+\frac{1}{j2n\pi}\left[1-(-1)^n\right] & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{1}{2}\left(1-\frac{1}{\pi}\right) & n=0\\ \mp j\left(\frac{1}{\pi}-\frac{1}{8}\right) & n=\pm1\\ \frac{1}{2\pi(n^2-1)} & 0\neq n=\text{even}\\ \frac{1}{jn\pi} & \pm1\neq n=\text{odd}\end{cases}$$

The expressions for the magnitude and phase spectra are given by

Magnitude Spectrum: $$|D_n| = \begin{cases} \frac{1}{2}\left(1-\frac{1}{\pi}\right) & n=0 \\ \frac{1}{\pi}-\frac{1}{8} & n=\pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \left|\frac{1}{n\pi}\right| & \pm 1 \neq n = odd \end{cases} = \begin{cases} \approx 0.3408 & n=0 \\ \approx 0.1933 & n=\pm 1 \\ \approx \frac{0.1592}{n^2-1} & 0 \neq n = even \\ \approx \frac{0.3183}{|n|} & \pm 1 \neq n = odd \end{cases}$$

Phase Spectrum: $$\angle D_n = \begin{cases} 0 & n = even \\ \angle(\mp j) & n = \pm 1 \\ \angle\left(\frac{1}{jn}\right) & \pm 1 \neq n = odd \end{cases} = \begin{cases} 0 & n = even \\ -\frac{\pi}{2} & n = odd, n>0 \\ \frac{\pi}{2} & n = odd, n<0 \end{cases}$$

The magnitude and phase spectra are shown in in row 5 of the subplots included in Fig. S4.11.

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Problem 4.13

In each case, we show that the exponential CTFS coefficients obtained directly from Eq. (4.44) are identical to those obtained from the trigonometric CTFS coefficients.

(a) From the solution of Problem P4.6(a), we know that

$$a_0 = \tfrac{1}{2},\ a_n = 0, \text{ and } b_n = \begin{cases} \frac{6}{n\pi} & n = odd \\ 0 & n = even \end{cases}.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x1(t)$ are given by

$$D_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n<0 \end{cases} = \begin{cases} a_0 & n=0 \\ -\frac{1}{2}jb_n & n>0 \\ \frac{1}{2}jb_{-n} & n<0 \end{cases} \qquad [\because\ a_n = a_{-n} = 0]$$

$$= \begin{cases} \frac{3}{2} & n=0 \\ 0 & n = even \\ -j\frac{3}{n\pi} & n=\text{odd}, n>0 \\ -j\frac{3}{n\pi} & n=\text{odd}, n<0 \end{cases} = \begin{cases} \frac{3}{2} & n=0 \\ 0 & n = even \\ \frac{3}{jn\pi} & n = odd \end{cases}$$

(c) From the solution of Problem P4.6(c), we know that

$$a_0 = \tfrac{1}{2},\quad a_n = 0,\quad \text{and}\quad b_n = \tfrac{1}{n\pi}.$$

Using Eq. (4.45), the exponential CTFS coefficients for $x3(t)$ are given by

$$D_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - jb_n) & n>0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n<0 \end{cases} = \begin{cases} a_0 & n=0 \\ -\frac{1}{2}jb_n & n>0 \\ \frac{1}{2}jb_{-n} & n<0 \end{cases} \qquad [\because\ a_n = a_{-n} = 0]$$

$$= \begin{cases} \frac{1}{2} & n=0 \\ -j\frac{1}{2n\pi} & n>0 \\ -j\frac{1}{2n\pi} & n<0 \end{cases} = \begin{cases} \frac{1}{2} & n=0 \\ \frac{-j}{2n\pi} & n \neq 0 \end{cases}$$

(e) From the solution of Problem P4.6(e), we know that

$$a_0 = \tfrac{1}{2} - \tfrac{1}{2\pi},\ a_n = \begin{cases} 0 & n = odd \\ \frac{1}{\pi(n^2-1)} & n = even \end{cases},\ \text{and}\ b_n = \begin{cases} 0 & n = even \\ \frac{2}{n\pi} - \frac{1}{4} & n = 1 \\ \frac{2}{n\pi} & 1 \neq n = odd \end{cases}$$

Using Eq. (4.45), the exponential CTFS coefficients for $x5(t)$ are given by

($n = 0$): $$D_0 = \tfrac{1}{2} - \tfrac{1}{2\pi}$$

($n = 1$): $$D_1 = \tfrac{1}{2}(a_1 - jb_1) = -\tfrac{j}{2}\left(\tfrac{2}{\pi} - \tfrac{1}{4}\right) = j\left(\tfrac{1}{8} - \tfrac{1}{\pi}\right)$$

($n = -1$): $$D_{-1} = \tfrac{1}{2}(a_1 + jb_1) = \tfrac{j}{2}\left(\tfrac{2}{\pi} - \tfrac{1}{4}\right) = -j\left(\tfrac{1}{8} - \tfrac{1}{\pi}\right)$$

($n > 1$): $$D_n = \tfrac{1}{2}(a_n - jb_n) = \begin{cases} -\frac{j}{n\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases} = \begin{cases} \frac{1}{jn\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases}$$

($n < -1$): $$D_n = \tfrac{1}{2}(a_{-n} + jb_{-n}) = \begin{cases} \frac{j}{2}\left(\frac{2}{-n\pi}\right) & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases} = \begin{cases} \frac{1}{jn\pi} & n = odd \\ \frac{1}{2\pi(n^2-1)} & n = even \end{cases}$$

Combining the above results, we obtain

$$D_n = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{\pi}\right) & n = 0 \\ \pm j\left(\frac{1}{8} - \frac{1}{\pi}\right) & n = \pm 1 \\ \frac{1}{2\pi(n^2-1)} & 0 \neq n = even \\ \frac{1}{jn\pi} & \pm 1 \neq n = odd. \end{cases}$$

Problem 5.2

(a) By definition,

$$X_1(\omega) = \int_0^{\pi} 3e^{-j\omega t}\,dt = 3\left[\frac{e^{-j\omega t}}{(-j\omega)}\right]_0^{\pi} = -\frac{3}{j\omega}\left[e^{-j\omega\pi} - 1\right] = -\frac{3}{j\omega}e^{-j\omega\pi/2}\left[e^{-j\omega\pi/2} - e^{j\omega\pi/2}\right]$$

$$= -\frac{3}{j\omega}e^{-j\omega\pi/2}\left[-2j\sin(\omega\pi/2)\right] = 6e^{-j\omega\pi/2}\left[\frac{\sin(\omega\pi/2)}{\omega}\right] = 6e^{-j\omega\pi/2}\left[\frac{1}{2/\pi} \times \frac{\sin(\omega\pi/2)}{\omega\pi/2}\right]$$

$$= 3\pi e^{-j\omega\pi/2}\operatorname{sinc}(\omega/2).$$

(b) By definition,

$$X_2(\omega) = \int_{-0.5T}^{0.5T} 0.5e^{-j\omega t}dt + \int_{0.5T}^{1.5T} e^{-j\omega t}dt = 0.5\left[\tfrac{e^{-j\omega t}}{(-j\omega)}\right]_{-0.5T}^{0.5T} + \left[\tfrac{e^{-j\omega t}}{(-j\omega)}\right]_{0.5T}^{1.5T}$$

$$= -\tfrac{0.5}{j\omega}\left[e^{-j0.5\omega T} - e^{j0.5\omega T}\right] - \tfrac{1}{j\omega}\left[e^{-j1.5\omega T} - e^{-j0.5\omega T}\right]$$

$$= -\tfrac{0.5}{j\omega}\left[-2j\sin(0.5\omega T)\right] - \tfrac{1}{j\omega}e^{-j\omega T}\left[-2j\sin(0.5\omega T)\right]$$

$$= \left[\tfrac{0.5T}{0.5T}\times\tfrac{\sin(0.5\omega T)}{\omega}\right] + 2e^{-j\omega T}\left[\tfrac{0.5T}{0.5T}\times\tfrac{\sin(0.5\omega T)}{\omega}\right]$$

$$= 0.5T\text{sinc}\left(\tfrac{0.5\omega \text{T}}{\pi}\right) + Te^{-j\omega T}\text{sinc}\left(\tfrac{0.5\omega \text{T}}{\pi}\right).$$

(c) By definition,

$$X_3(\omega) = \int_0^T \left(1-\tfrac{t}{T}\right)e^{-j\omega t}dt = \left[\left(1-\tfrac{t}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)} - \left(-\tfrac{1}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)^2}\right]_0^T$$

$$= \left[0 - \left(-\tfrac{1}{T}\right)\tfrac{e^{-j\omega T}}{(-j\omega)^2}\right] - \left[\tfrac{1}{(-j\omega)} - \left(-\tfrac{1}{T}\right)\tfrac{1}{(-j\omega)^2}\right]$$

$$= -\tfrac{1}{\omega^2 T}e^{-j\omega T} + \tfrac{1}{j\omega} + \tfrac{1}{\omega^2 T} = \tfrac{1}{j\omega} + \tfrac{1}{\omega^2 T}\left(1 - e^{-j\omega T}\right).$$

For $\omega = 0$, $$X_3(\omega) = \int_0^T \left(1-\tfrac{t}{T}\right)dt = \left[-\tfrac{T}{2}\left(1-\tfrac{t}{T}\right)^2\right]_0^T = 0 + \tfrac{T}{2} = \tfrac{T}{2}.$$

(d) By definition,

$$X_4(\omega) = \int_{-T}^{0}\left(1+\tfrac{t}{T}\right)e^{-j\omega t}dt + \int_0^T\left(1-\tfrac{t}{T}\right)e^{-j\omega t}dt$$

$$= \left[\left(1+\tfrac{t}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)} - \left(\tfrac{1}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)^2}\right]_{-T}^{0} + \left[\left(1-\tfrac{t}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)} - \left(-\tfrac{1}{T}\right)\tfrac{e^{-j\omega t}}{(-j\omega)^2}\right]_0^T$$

$$= \left[\tfrac{1}{(-j\omega)} - \left(\tfrac{1}{T}\right)\tfrac{1}{(-j\omega)^2} - 0 + \left(\tfrac{1}{T}\right)\tfrac{e^{j\omega T}}{(-j\omega)^2}\right] + \left[0 - \left(-\tfrac{1}{T}\right)\tfrac{e^{-j\omega T}}{(-j\omega)^2} - \tfrac{1}{(-j\omega)} + \left(-\tfrac{1}{T}\right)\tfrac{1}{(-j\omega)^2}\right]$$

$$= \tfrac{2}{\omega^2 T}\left[1 - \cos(\omega T)\right] = \tfrac{2\times 2\sin^2(0.5\omega T)}{\omega^2 T} = \tfrac{4}{1/(0.5^2 T)}\times\tfrac{\sin^2(0.5\omega T)}{(0.5\omega T)^2} = T\text{sinc}^2\left(\tfrac{0.5\omega T}{\pi}\right).$$

(e) By definition,

$$X_5(\omega) = \int_0^T \left[1 - 0.5\sin\left(\tfrac{\pi t}{T}\right)\right]e^{-j\omega t}dt = \underbrace{\int_0^T e^{-j\omega t}dt}_{=A} - \underbrace{0.5\int_0^T \sin\left(\tfrac{\pi t}{T}\right)e^{-j\omega t}dt}_{=B}$$

We consider different cases for the above integral.

Case I: (ω = 0)

$$X_5(0)=\int_{\infty}^{-\infty} x(t)dt=\int_{-\infty}^{\infty}\left[1-0.5\sin\left(\tfrac{\pi t}{T}\right)\right]dt=\int_0^T dt-0.5\int_0^T \sin\left(\tfrac{\pi t}{T}\right)dt$$

$$=T+\frac{0.5}{\pi/T}\left[\cos(\tfrac{\pi t}{T})\right]_0^T=T+\frac{T}{2\pi}\left[\cos(\pi)-\cos(0)\right]=T-\frac{T}{\pi}=T(1-\tfrac{1}{\pi})$$

Case II: (ω ≠ 0, ω ≠ π/*T*):

$$A=\int_0^T e^{-j\omega t}dt=\frac{1}{-j\omega}\left[e^{-j\omega t}\right]_0^T=\frac{1}{-j\omega}\left[e^{-j\omega T}-1\right]=\frac{1}{j\omega}\left[1-e^{-j\omega T}\right] \qquad [\omega\neq 0]$$

$$B=0.5\left[\frac{e^{-j\omega t}}{\frac{\pi^2}{T^2}-\omega^2}\left\{-j\omega\sin\left(\tfrac{\pi t}{T}\right)-\tfrac{\pi}{T}\cos\left(\tfrac{\pi t}{T}\right)\right\}\right]_0^T \qquad \text{for } \omega\neq 0,\pm\tfrac{\pi}{T}$$

$$=\frac{0.5T^2}{\pi^2-\omega^2T^2}\left[e^{-j\omega t}\left\{\underbrace{j\omega\sin\left(\tfrac{\pi t}{T}\right)}_{=0\ at\ t=0,T}+\tfrac{\pi}{T}\cos\left(\tfrac{\pi t}{T}\right)\right\}\right]_0^T$$

$$=\frac{0.5T^2}{\pi^2-\omega^2T^2}\left[-\tfrac{\pi}{T}e^{-j\omega T}-\tfrac{\pi}{T}\right]=\frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right]$$

Case III: (ω = π/*T*):

$$B=0.5\int_0^T\sin\left(\tfrac{\pi t}{T}\right)e^{-j\omega t}dt=\frac{0.5}{2j}\int_0^T\left[e^{j\frac{\pi t}{T}}-e^{-j\frac{\pi t}{T}}\right]e^{-j\omega t}dt=\frac{0.5}{2j}\int_0^T\left[e^{-j(\omega-\frac{\pi}{T})t}-e^{-j(\omega+\frac{\pi}{T})t}\right]dt$$

$$=\begin{cases}\frac{0.5}{2j}\int_0^T\left[1-e^{-j\frac{2\pi t}{T}}\right]dt & \omega=\frac{\pi}{T}\\ -\frac{0.5}{2j}\int_0^T\left[1-e^{j\frac{2\pi t}{T}}\right]dt & \omega=-\frac{\pi}{T}\end{cases} \qquad \left[As\ e^{\pm\frac{j2\pi t}{T}}\ is\ periodic\ with\ period\ T,\ \int_0^T e^{\pm\frac{j2\pi t}{T}}dt=0\right]$$

$$=\pm\frac{0.5}{2j}\left[t\right]_0^T=\pm\frac{0.5T}{2j}$$

Combining, the above results, the CTFT can be expressed as

$$X_5(\omega)=\begin{cases}T(1-\frac{1}{\pi}) & \omega=0\\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right]\mathrm{m}\frac{0.5T}{2j} & \omega=\pm\frac{\pi}{T}\\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right]-\frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right] & otherwise\end{cases}$$

$$=\begin{cases}T(1-\frac{1}{\pi}) & \omega=0\\ \pm\frac{2T}{j\pi}\mathrm{m}\frac{T}{4j} & \omega=\pm\frac{\pi}{T}\\ \frac{1}{j\omega}\left[1-e^{-j\omega T}\right]-\frac{0.5\pi T}{\pi^2-\omega^2T^2}\left[1+e^{-j\omega T}\right] & otherwise\end{cases}$$

∎

Problem 5.4

(a) The partial fraction expansion is given by

$$X_1(\omega) = \frac{(1+j\omega)}{(2+j\omega)(3+j\omega)} \equiv \frac{-1}{(2+j\omega)} + \frac{2}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_1(t) = -e^{-2t}u(t) + 2e^{-3t}u(t).$$

(b) The partial fraction expansion is given by

$$X_2(\omega) = \frac{1}{(1+j\omega)(2+j\omega)(3+j\omega)} \equiv \frac{0.5}{(1+j\omega)} + \frac{-1}{(2+j\omega)} + \frac{0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_2(t) = 0.5e^{-t}u(t) - e^{-2t}u(t) + 0.5e^{-3t}u(t).$$

(c) The partial fraction expansion is given by

$$X_3(\omega) = \frac{1}{(1+j\omega)(2+j\omega)^2(3+j\omega)} \equiv \frac{0.5}{(1+j\omega)} + \frac{0}{(2+j\omega)} + \frac{-1}{(2+j\omega)^2} + \frac{-0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$x_3(t) = 0.5e^{-t}u(t) - te^{-2t}u(t) + 0.5e^{-3t}u(t).$$

Problem 5.9

(a) Applying the linearity property,

$$X_1(\omega) = \Im\{5 + 3\cos(10t) - 7e^{-2t}\sin(3t)u(t)\} = 5\Im\{1\} + 3\Im\{\cos(10t)\} - 7\Im\{e^{-2t}\sin(3t)u(t)\}.$$

By selecting the appropriate CTFT pairs from Table 5.2, we get

$$X_1(\omega) = 10\delta(\omega)\Im\{1\} + 3\pi\delta(\omega-10) + 3\pi\delta(\omega-10) - \frac{21}{(2+j\omega)^2 + 3^2}.$$

(b) Entry (8) of Table 5.2 provides the CTFT pair

$$\mathrm{sgn}(t) \xleftrightarrow{\text{CTFT}} \frac{2}{j\omega}.$$

Using the duality property, $\frac{2}{jt} \xleftrightarrow{\text{CTFT}} 2\pi\,\mathrm{sgn}(-\omega),$

or, $\frac{1}{\pi t} \xleftrightarrow{\text{CTFT}} -j\,\mathrm{sgn}(\omega).$

(c) Entry (7) of Table 5.2 provides the CTFT pair

$$e^{-4|t|} \xleftrightarrow{\text{CTFT}} \frac{8}{4+j\omega}.$$

Using the time shifting property, $e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} \frac{8}{4+j\omega}e^{-j5\omega}.$

Using the frequency differentiation property,

$$t^2 e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} (j)^2 \frac{d^2}{d\omega^2}\left\{e^{-j5\omega}\frac{8}{4+j\omega}\right\}$$

or, $$t^2 e^{-4|t-5|} \xleftrightarrow{\text{CTFT}} 200e^{-j5\omega}\frac{1}{4+j\omega} + 16e^{-j5\omega}\frac{1}{(4+j\omega)^3}.$$

(d) Entry (17) of Table 5.2 provides the CTFT pair

$$3\,\text{sinc}(3t) = 3\frac{\sin(3\pi t)}{3\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\frac{\omega}{6\pi}\right)$$

and $$5\,\text{sinc}(5t) = 5\frac{\sin(5\pi t)}{5\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\frac{\omega}{10\pi}\right)$$

Using the multiplication property

$$\pi^2 \times \frac{\sin(3\pi t)}{\pi t} \times \frac{\sin(5\pi t)}{\pi t} \xleftrightarrow{\text{CTFT}} \frac{\pi^2}{2\pi}\left[\text{rect}\left(\frac{\omega}{6\pi}\right) * \text{rect}\left(\frac{\omega}{10\pi}\right)\right]$$

or, $$\frac{\sin(3\pi t)\sin(5\pi t)}{t^2} \xleftrightarrow{\text{CTFT}} \frac{\pi}{2}\left[\text{rect}\left(\frac{\omega}{6\pi}\right) * \text{rect}\left(\frac{\omega}{10\pi}\right)\right],$$

or, $$5\frac{\sin(3\pi t)\sin(5\pi t)}{t^2} \xleftrightarrow{\text{CTFT}} \frac{5\pi}{2}\left[\text{rect}\left(\frac{\omega}{6\pi}\right) * \text{rect}\left(\frac{\omega}{10\pi}\right)\right],$$

where * is the convolution operation.

(e) Entry (17) of Table 5.2 provides the CTFT pair

$$3\,\text{sinc}(3t) = 3\frac{\sin(3\pi t)}{3\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\frac{\omega}{6\pi}\right)$$

and $$4\,\text{sinc}(4t) = 4\frac{\sin(4\pi t)}{4\pi t} \xleftrightarrow{\text{CTFT}} \text{rect}\left(\frac{\omega}{8\pi}\right).$$

Using the time differentiation property,

$$\frac{1}{\pi}\frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} (j\omega)\text{rect}\left(\frac{\omega}{8\pi}\right).$$

Using the convolution property

$$\pi^2 \times \frac{\sin(3\pi t)}{\pi t} * \frac{1}{\pi}\frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} \frac{\pi^2}{2\pi}\left[\text{rect}\left(\frac{\omega}{6\pi}\right) \times j\omega\text{rect}\left(\frac{\omega}{8\pi}\right)\right]$$

or, $$\frac{\sin(3\pi t)}{t} * \frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} \frac{\pi}{2}\left[\text{rect}\left(\frac{\omega}{6\pi}\right) \times j\omega\text{rect}\left(\frac{\omega}{8\pi}\right)\right],$$

or, $$4\frac{\sin(3\pi t)}{t} * \frac{d}{dt}\frac{\sin(4\pi t)}{t} \xleftrightarrow{\text{CTFT}} j2\pi\,\text{rect}\left(\frac{\omega}{6\pi}\right).$$

Problem 5.15

(a) Using the time scaling property, $x(2t) \xleftrightarrow{\text{CTFT}} \frac{1}{2}X\left(\frac{\omega}{2}\right)$.

Using the frequency shifting property, $e^{-j5t}x(2t) \xleftrightarrow{\text{CTFT}} \frac{1}{2}X\left(\frac{\omega+5}{2}\right)$.

Substituting the value of $X(\omega)$, we obtain

$$\Im\left\{e^{-j5t}x(2t)\right\} = \tfrac{1}{2}\begin{cases}1-\frac{|\omega+5|}{3} & |\omega+5|\le 3\\ 0 & \text{elsewhere}\end{cases}$$

$$= \begin{cases}\frac{\omega+11}{12} & -11\le\omega\le -5\\ \frac{1-\omega}{12} & -5\le\omega\le -1\\ 0 & \text{elsewhere.}\end{cases}$$

(b) Using the frequency differentiation property,

$$(jt)^2 x(t) \xleftrightarrow{\text{CTFT}} \frac{d^2X}{d\omega^2},$$

or,

$$t^2 x(t) \xleftrightarrow{\text{CTFT}} -\frac{d^2X}{d\omega^2}.$$

The CTFT of $t^2\, x(t)$ is given by

$$F\left\{t^2 x(t)\right\} = -\frac{d^2}{d\omega^2}\left[\Delta\left(\frac{\omega}{3}\right)\right] = -\frac{d}{d\omega}\left[\text{rect}\left(\frac{\omega}{3}\right)\right] = -[\delta(\omega+3)-\delta(\omega-3)] = [\delta(\omega-3)-\delta(\omega+3)].$$

(c) Express $(t+5)\frac{dx}{dt} = t\frac{dx}{dt} + 5\frac{dx}{dt}$.

Using the time differentiation property, the CTFT of $\frac{dx}{dt}$ is given by

$$\frac{dx}{dt} \xleftrightarrow{\text{CTFT}} j\omega X(\omega).$$

Applying the frequency differentiation property to the above CTFT pair, gives

$$t\frac{dx}{dt} \xleftrightarrow{\text{CTFT}} j\frac{d}{d\omega}\left[j\omega X(\omega)\right] = -X(\omega) - \omega\frac{dX}{d\omega}.$$

The CTFT of $(t+5)\frac{dx}{dt}$ is given by

$$\Im\left\{(t+5)\frac{dx}{dt}\right\} = -X(\omega) - \omega\frac{dX}{d\omega} + 5j\omega X(\omega).$$

Substituting the value of $X(\omega)$, we obtain

$$\Im\left\{(t+5)\frac{dx}{dt}\right\} = \begin{cases} j5\omega\left(1-\frac{\omega}{3}\right) - \left(1-\frac{2\omega}{3}\right) & 0\le\omega\le 3\\ j5\omega\left(1+\frac{\omega}{3}\right) - \left(1+\frac{2\omega}{3}\right) & -3\le\omega\le 0\\ 0 & \text{elsewhere.}\end{cases}$$

(d) Using the time multiplication property,

$$x(t)\cdot x(t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}\left[X(\omega) * X(\omega)\right],$$

which implies that

$$F\{x(t)\cdot x(t)\} = \frac{1}{2\pi}\left[\Delta\left(\frac{\omega}{3}\right) * \Delta\left(\frac{\omega}{3}\right)\right].$$

(e) Using the time convolution property,

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which reduces to

$$F\{x(t)*x(t)\}=\begin{cases}\left[1-\frac{|\omega|}{3}\right]^2 & |\omega|\le 3\\ 0 & \text{elsewhere}\end{cases} = \begin{cases}1+\frac{\omega^2}{9}-\frac{2|\omega|}{3} & |\omega|\le 3\\ 0 & \text{elsewhere.}\end{cases}$$

(f) Using the time multiplication property,

$$x(t)\cdot\cos\omega_0 t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2\pi}X(\omega)*\pi\delta(\omega-\omega_0)+\tfrac{1}{2\pi}X(\omega)*\pi\delta(\omega+\omega_0),$$

or,
$$x(t)\cdot\cos\omega_0 t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X(\omega-\omega_0)+\tfrac{1}{2}X(\omega+\omega_0)$$

Case I: For $\omega_0 = 3/2$, we obtain

$$x(t)\cdot\cos(3t/2) \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X\left(\omega-\tfrac{3}{2}\right)+\tfrac{1}{2}X\left(\omega+\tfrac{3}{2}\right).$$

The two replicas overlap over $(-3/2 \le \omega < 3/2)$, therefore,

$$F\{x(t)\cos(3t/2)\}=\begin{cases}\frac{1}{2}+\frac{\omega+3/2}{6} & -\frac{9}{2}\le\omega\le-\frac{3}{2}\\ 1 & -\frac{3}{2}\le\omega\le\frac{3}{2}\\ \frac{1}{2}+\frac{\omega-3/2}{6} & \frac{3}{2}\le\omega\le\frac{9}{2}\\ 0 & \text{elsewhere.}\end{cases}$$

Case II: For $\omega_0 = 3$, we obtain

$$x(t)\cdot\cos 3t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X(\omega-3)+\tfrac{1}{2}X(\omega+3).$$

Since there is no overlap between the two shifted replicas,

$$F\{x(t)\cos 3t\}=\tfrac{1}{2}\begin{cases}1-\frac{|\omega+3|}{3} & |\omega+3|\le 3\\ 1-\frac{|\omega-3|}{3} & |\omega-3|\le 3\\ 0 & \text{elsewhere.}\end{cases}$$

or,
$$F\{x(t)\cos 3t\}=\begin{cases}\frac{1}{2}-\frac{|\omega+3|}{6} & -6\le\omega<0\\ \frac{1}{2}-\frac{|\omega-3|}{6} & 0\le\omega<6\\ 0 & \text{elsewhere.}\end{cases}$$

Case III: For $\omega_0 = 6$, we obtain

$$x(t)\cdot\cos 6t \xleftrightarrow{\text{CTFT}} \tfrac{1}{2}X(\omega-6)+\tfrac{1}{2}X(\omega+6).$$

Since there is no overlap between the two shifted replicas,

$$F\{x(t)\cos 3t\}=\tfrac{1}{2}\begin{cases}1-\frac{|\omega+6|}{3} & |\omega+6|\le 3\\ 1-\frac{|\omega-6|}{3} & |\omega-6|\le 3\\ 0 & \text{elsewhere.}\end{cases}$$

or,
$$F\{x(t)\cos 3t\} = \begin{cases} \frac{1}{2} - \frac{|\omega+6|}{6} & -9 \le \omega < -3 \\ \frac{1}{2} - \frac{|\omega-6|}{6} & 3 \le \omega < 9 \\ 0 & \text{elsewhere.} \end{cases} \qquad \blacksquare$$

Problem 5.20

(a) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^3 Y(\omega) + 5(j\omega)^2 Y(\omega) + 11(j\omega)Y(\omega) + 6Y(\omega) = X(\omega),$$

or,
$$\left((j\omega)^3 + 5(j\omega)^2 + 11(j\omega) + 6\right)Y(\omega) = X(\omega),$$

or,
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^3 + 5(j\omega)^2 + 11(j\omega) + 6}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega) = \frac{1}{(1+j\omega)(2+j\omega)(3+j\omega)} \equiv \frac{0.5}{(1+j\omega)} + \frac{-1}{(2+j\omega)} + \frac{0.5}{(3+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t) = 0.5e^{-t}u(t) - e^{-2t}u(t) + 0.5e^{-3t}u(t).$$

(b) Calculating the CTFT of both sides and applying the time differential property, yields

$$(j\omega)^2 Y(\omega) + 3(j\omega)Y(\omega) + 2Y(\omega) = X(\omega),$$

or,
$$\left((j\omega)^2 + 3(j\omega) + 2\right)Y(\omega) = X(\omega),$$

or,
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 3(j\omega) + 2}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega) = \frac{1}{(j\omega)^2 + 3(j\omega) + 2} \equiv \frac{1}{(1+j\omega)} - \frac{1}{(2+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t) = e^{-t}u(t) - e^{-2t}u(t).$$

(c) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^2 Y(\omega) + 2(j\omega)Y(\omega) + Y(\omega) = X(\omega),$$

or,
$$\left((j\omega)^2 + 1(j\omega) + 1\right)Y(\omega) = X(\omega),$$

or,
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + 2(j\omega) + 1}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega)=\frac{1}{(1+j\omega)^2}$$

Calculating the inverse CTFT, we obtain

$$h(t)=te^{-t}u(t).$$

(d) Calculating the CTFT of both sides and applying the time differentiation property, yields

$$(j\omega)^2Y(\omega)+6(j\omega)Y(\omega)+8Y(\omega)=(j\omega)X(\omega)+4X(\omega),$$

or,

$$\left((j\omega)^2+6(j\omega)+8\right)Y(\omega)=((j\omega)+4)X(\omega),$$

or,

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{(j\omega)+4}{(j\omega)^2+6(j\omega)+8}=\frac{1}{2+j\omega}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which is given by

$$h(t)=e^{-2t}u(t).$$

(e) Calculating the CTFT of both sides and applying the time differential property, yields

$$(j\omega)^3Y(\omega)+8(j\omega)^2Y(\omega)+19(j\omega)Y(\omega)+12Y(\omega)=X(\omega),$$

or,

$$\left((j\omega)^3+8(j\omega)^2+19(j\omega)+12\right)Y(\omega)=X(\omega),$$

or,

$$H(\omega)=\frac{Y(\omega)}{X(\omega)}=\frac{1}{(j\omega)^3+8(j\omega)^2+19(j\omega)+12}.$$

The impulse response $h(t)$ can be obtained by calculating the inverse CTFT of $H(\omega)$, which can be expressed as

$$H(\omega)=\frac{1}{(j\omega)^3+8(j\omega)^2+19(j\omega)+12}\equiv\frac{1/6}{(1+j\omega)}+\frac{-1/2}{(3+j\omega)}+\frac{1/3}{(4+j\omega)}$$

Calculating the inverse CTFT, we obtain

$$h(t)=\tfrac{1}{6}e^{-t}u(t)-\tfrac{1}{2}e^{-3t}u(t)+\tfrac{1}{3}e^{-4t}u(t).$$

▌

Problem 5.29

(a) In Example 3.6, it was shown that

$$y(t)=e^{-t}u(t)*e^{-2t}u(t)=\left[e^{-t}-e^{-2t}\right]u(t).$$

(b) From Table 5.2, the CTFT of $x(t)$ and $h(t)$ are obtained as

$$X(\omega)=\tfrac{1}{1+j\omega},\quad \text{and} \quad H(\omega)=\tfrac{1}{2+j\omega}.$$

The CTFT of the output is then given by

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(1+j\omega)}\frac{1}{(2+j\omega)} = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}.$$

Calculating the inverse CTFT results in the output signal

$$y(t) = \left[e^{-t} - e^{-2t}\right] u(t).$$

(c) As $H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2+j\omega}$, the Fourier-domain input-output relationship can be expressed as

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega).$$

Calculating the inverse CTFT of both sides results in the following differential equation

$$\frac{dy}{dt} + 2y(t) = x(t).$$

The output can be obtained by solving the differential equation with input $x(t) = e^{-t}u(t)$ and zero initial conditions $y(0^-) = 0$.

Zero-input Response: Due to zero initial condition, the zero-input response is $y_{zi}(t) = 0$.

Zero-state Response: The characteristics equation is given by $(s + 2) = 0$ resulting in a single pole at $s = -2$. The homogenous component of the zero-state response is given by

$$y_{zs}^h(t) = Ae^{-2t}.$$

Since the input $x(t) = \exp(-t)\, u(t)$, the particular solution is of the form

$$y_{zs}^p(t) = Ke^{-t} \text{ for } t \geq 0.$$

Inserting the particular solution in the differential equation results in $K = 1$. Therefore,

$$y_{zs}^p(t) = e^{-t}u(t).$$

The overall zero-state response is, therefore, given by

$$y_{zs}(t) = Ae^{-2t} + e^{-t}$$

for $t \geq 0$. To determine the value of A, we insert the initial condition $y(0^-) = 0$ giving

$$A + 1 = 0 \Rightarrow A = -1$$

or, $A = -1$. The zero state response is given by

$$y_{zs}(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

Total Response: By adding the zero-input and zero-state responses, the overall output is given by

$$y(t) = \underbrace{y_{zi}(t)}_{=0} + y_{zs}(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

It is observed that Methods (a) – (c) yield the same result. ▌

Problem 5.31

(a) The magnitude spectra of the two systems are calculated below

$$|H_1(\omega)| = \frac{\sqrt{400+\omega^2}}{\sqrt{400+\omega^2}} = 1$$

$$|H_2(\omega)| = \begin{cases} 1 & |\omega| \geq 20 \\ 0 & \text{elsewhere.} \end{cases}$$

The magnitude spectra are plotted in Fig. S5.31). From Fig. S5.31(a), we observe that the magnitude $|H_1(\omega)|$ is 1 at all frequencies. Therefore, System $H_1(\omega)$ is an all pass filter.

From Fig. S5.31(b), we observe that the magnitude $|H_2(\omega)|$ is zero at frequencies below 20 radians/s. At frequencies above 20 radians/s, the magnitude is 1. Therefore, System $H_2(\omega)$ is a highpass filter.

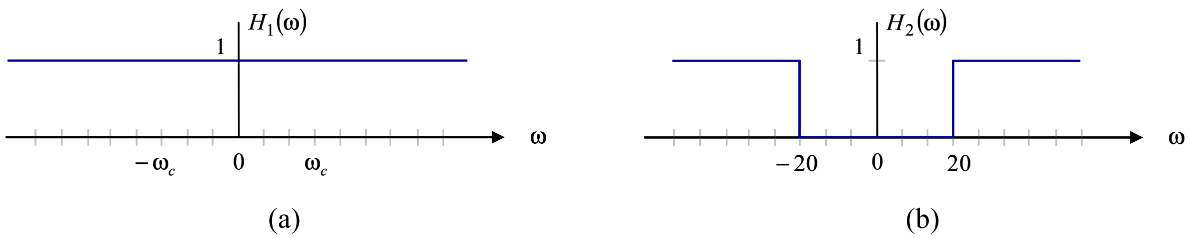

Fig. S5.31: Magnitude Spectra for Problem 5.31.

(b) Calculating the inverse CTFT, the impulse response of the two systems is given by

$$h_1(t) = \Im^{-1}\left\{\frac{40-20-j\omega}{20+j\omega}\right\} = \Im^{-1}\left\{\frac{40}{20+j\omega}\right\} - \Im^{-1}\{1\} = 40e^{-20t}u(t) - \delta(t).$$

$$h_2(t) = \Im^{-1}\left\{1-\text{rect}(\tfrac{\omega}{40})\right\} = \Im^{-1}\{1\} - \Im^{-1}\left\{\text{rect}(\tfrac{\omega}{40})\right\} = \delta(t) - \tfrac{20}{\pi}\text{sinc}(\tfrac{20t}{\pi}).$$ ▌

Problem 5.32

The transfer functions for the three LTIC systems are given by

System (a): $$H_1(\omega) = \frac{2}{(1+j\omega)^2}.$$

System (b): $$H_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}.$$

System (c): $$H_3(\omega) = -2 + \frac{5}{2+j\omega} = \frac{1-j2\omega}{2+j\omega}$$

The following Matlab code generates the magnitude and phase spectra of the three LTIC systems.

```
%MATLAB Program for Problem P5.32
%System (a)
clear;                              % clear the MATLAB environment
num_coeff = [2];                    % NUM coeffs. in decreasing powers of s
denom_coeff = [1 2 1];              % DEN coeffs. in decreasing powers of s
```

```
sys = tf(num_coeff,denom_coeff);      % specify the transfer function
figure(1)
bode(sys,{0.02,100}); grid;           % sketch the Bode plots
title('Bode Plot for System-1')
%System (b)
clear;                                % clear the MATLAB environment
num_coeff = [1];                      % NUM coeffs. in decreasing powers of s
denom_coeff = [1 0];                  % DEN coeffs. in decreasing powers of s
sys = tf(num_coeff,denom_coeff);      % specify the transfer function
figure(2)
bode(sys,{0.02,100}); grid;           % sketch the Bode plots
title('Bode Plot for System-2')
%System (vc)
clear;                                % clear the MATLAB environment
num_coeff = [-2 1];                   % NUM coeffs. in decreasing powers of s
denom_coeff = [1 2];                  % DEN coeffs. in decreasing powers of s
sys = tf(num_coeff,denom_coeff);      % specify the transfer function
figure(3)
bode(sys,{0.02,100}); grid;           % sketch the Bode plots
title('Bode Plot for System-3')
```

The resulting Bode plots are shown in Fig. S5.32.

Calculating Output:

System (a): Using the modulation property, the output of system (a) is given by

$$Y_1(\omega) = \frac{2}{(1+j\omega)^2} \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] = 2\pi\left(\tfrac{1}{(1+j1)^2}\delta(\omega-1)+\tfrac{1}{(1-j1)^2}\delta(\omega+1)\right)$$
$$= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right].$$

Calculating the inverse CTFT, we obtain

$$y_1(t) = \sin t.$$

System (b): Using the modulation property, the output of system (a) is given by

$$Y_2(\omega) = \left[\pi\delta(\omega)+\frac{1}{j\omega}\right] \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] = \pi\left[\tfrac{1}{j}\delta(\omega-1)+\tfrac{1}{-j}\delta(\omega+1)\right]$$
$$= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right].$$

Calculating the inverse CTFT, we obtain

$$y_2(t) = \sin t.$$

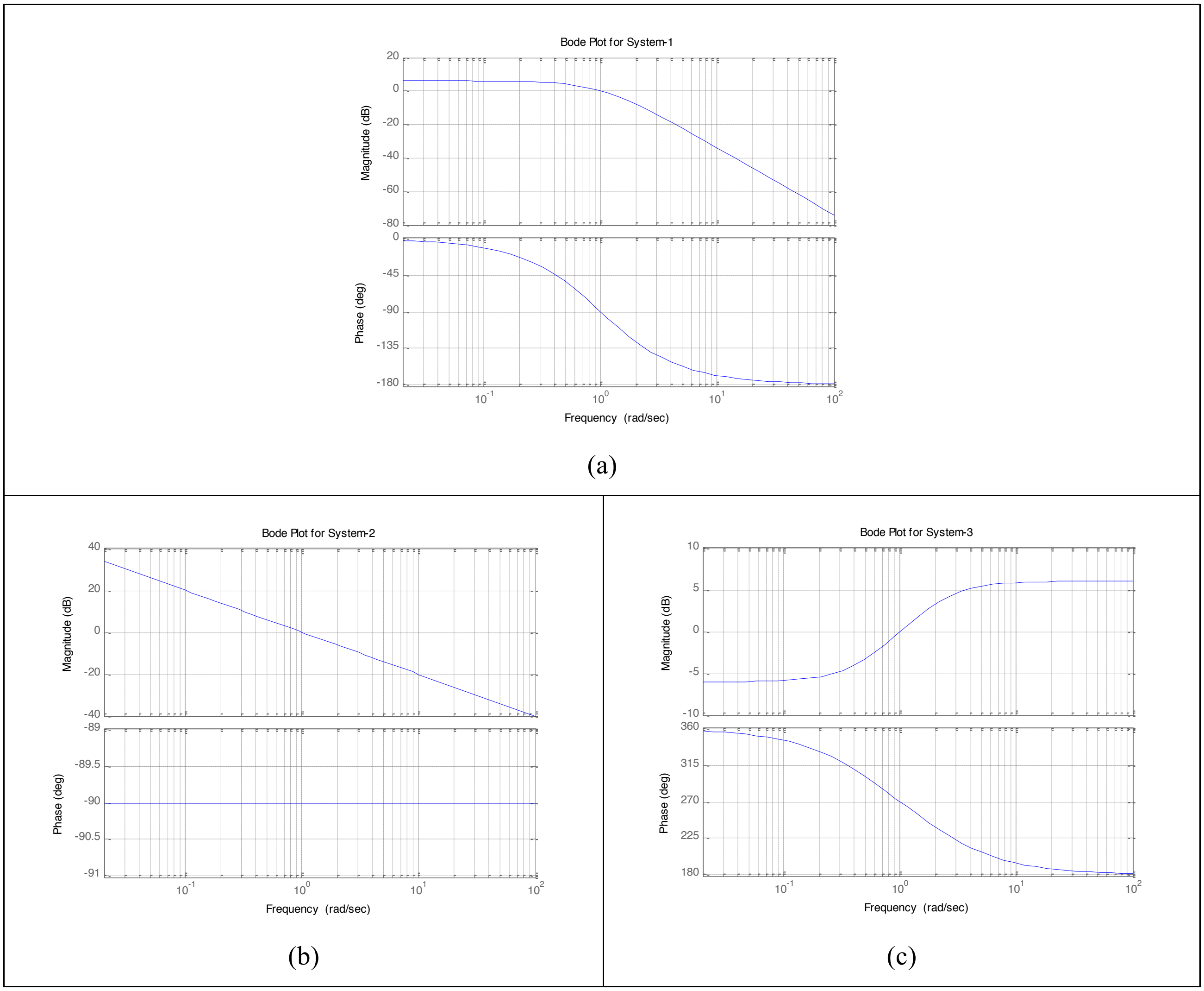

Figure S5.32. Magnitude and phase spectra for systems in Problem 5.32.

System (c): Using the modulation property, the output of system (a) is given by

$$\begin{aligned} Y_2(\omega) &= \frac{1-j2\omega}{2+j\omega} \times \pi\left[\delta(\omega-1)+\delta(\omega+1)\right] \\ &= \pi\left[\tfrac{1-j2}{2+j}\,\delta(\omega-1)+\tfrac{1+j2}{2-j}\,\delta(\omega+1)\right] \\ &= -j\pi\left[\delta(\omega-1)-\delta(\omega+1)\right]. \end{aligned}$$

Calculating the inverse CTFT, we obtain

$$y_3(t) = \sin t\,.$$

To explain why the three systems produce the same output for input $x(t) = \cos t$, consider Eq. (5.77), which for $\omega_0 = 1$ is given by

$$\cos(t) \xrightarrow{\text{Hermitian Symmetric } H(\omega)} |H(1)|\cos(\omega_0 t + \angle H(1)).$$

In other words, the output for $x(t) = \cos(t)$ depends only on the magnitude and phase of the system at $\omega = 1$. For the three systems, we note that

$$|H_1(1)| = |H_2(1)| = |H_3(1)| = 1$$

and

$$< H_1(1) = H_2(1) = H_3(1) = -\tfrac{\pi}{2}.$$

Since the magnitudes and phases of the three systems at $\omega = 1$ are the same, the three systems produce the same output $y(t) = \sin t$ for $x(t) = \cos(t)$. $\blacksquare$