No. 3
Fourier Analysis of LTIC Systems

Prof. Hui Jiang
Department of Electrical Engineering and Computer Science
Lassonde School of Engineering, York University

LTIC systems

- Analyze LTIC system from different perspectives
  - Fourier analysis (in frequency domain)
- Signals can be decomposed based on any complete set of orthogonal signals.

Fourier Analysis (Frequency Analysis)

Input signals

$x_1(t) = k_1 e^{j\omega_1 t}$

$x_2(t) = k_1 \cos(\omega_1 t)$

$x_3(t) = k_1 \sin(\omega_1 t)$

→ $h(t)$ (LTI) →

Output signals

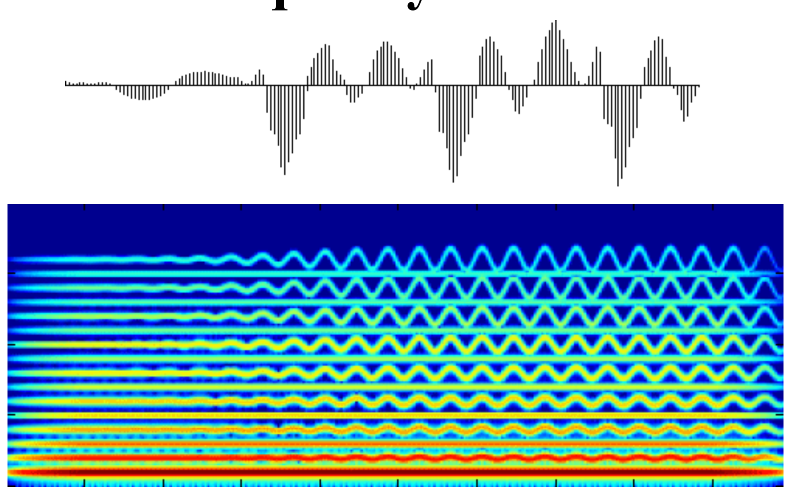
$x_1(t) = A_1 k_1 e^{j(\omega_1 t + \phi_1)}$

$x_2(t) = A_1 k_1 \cos(\omega_1 t + \phi_1)$

$x_3(t) = A_1 k_1 \sin(\omega_1 t + \phi_1)$

15-02-25 3

CT Signal Decomposition in Frequency domain



15-02-25 4

Fourier Analysis

- Decompose signals in frequency domain.
- Base orthogonal signals are single frequency components:
  - Complex exponential signals
  - Sinusoidal signals
- Why use these base signals?
  - Complex exponential (sinusoidal) signals are eigen functions of LTIC systems.

Fourier Analysis

- Periodic CT signals:
  - Trigonometric CT Fourier Series (CTFS)
  - Exponential CT Fourier Series (CTFS)
- Aperiodic CT signals:
  - CT Fourier Transform (CTFT)

Trigonometric Fourier Series

- Fourier theorem:

An arbitrary periodic function $x(t)$ with fundamental period T_0 can be expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of $x(t)$ and coefficients a_0, a_n, and b_n (the trigonometric CTFS coefficients) are given by

$$a_0 = \frac{1}{T_0}\int_{\langle T_0 \rangle} x(t)dt$$

$$a_n = \frac{2}{T_0}\int_{\langle T_0 \rangle} x(t)\cos(n\omega_0 t)dt$$

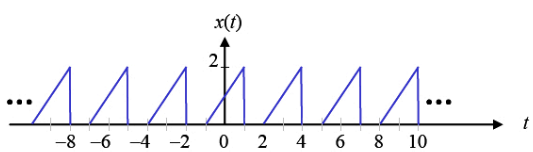
$$b_n = \frac{2}{T_0}\int_{\langle T_0 \rangle} x(t)\sin(n\omega_0 t)dt$$

15-02-25 7

Trigonometric CTFS: example

- Sawtooth period signal:

$$x(t) = \begin{cases} t+1 & -1 \le t \le 1 \\ 0 & 1 < t < 2 \end{cases}$$



Solution:

$$a_0 = \tfrac{2}{3}$$

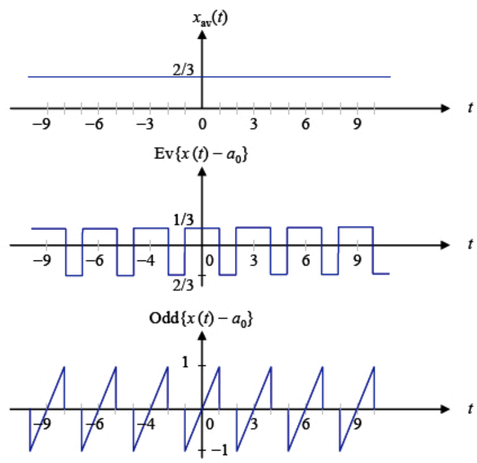
$$a_n = \begin{cases} 0 & n = 3k \\ \frac{\sqrt{3}}{n\pi} & n = 3k+1 \\ -\frac{\sqrt{3}}{n\pi} & n = 3k+2 \end{cases}$$

$$b_n = -\frac{4\cos(n\omega_0)}{3n\omega_0} + \frac{4\sin(n\omega_0)}{3(n\omega_0)^2}$$

15-02-25 8

Sawtooth Signals

$$x(t) = \underbrace{\tfrac{2}{3}}_{x_{av}(t)} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\left(\tfrac{2n\pi}{3} t\right)}_{\mathrm{Ev}\{x(t)-a_0\}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\tfrac{2n\pi}{3} t\right)}_{\mathrm{Odd}\{x(t)-a_0\}}$$



15-02-25 9

Symmetric CT Signals

1. If $x(t)$ is zero-mean, then $a_0 = 0$.
2. If $x(t)$ is an even function, then $b_n = 0$, for all n.
3. If $x(t)$ is an odd function, then $a_0 = a_n = 0$, for all n.
4. If $x(t)$ is a real function, then the trigonometric CTFS coefficients a_0, a_n, and b_n are also real valued for all n.
5. If $g(t) = x(t) + c$ (where c is a constant) then the trigonometric CTFS coefficients $\{a_0^g, a_n^g, b_n^g\}$ of $g(t)$ are given by

DC coefficient: $a_0^g = a_0^x + c$

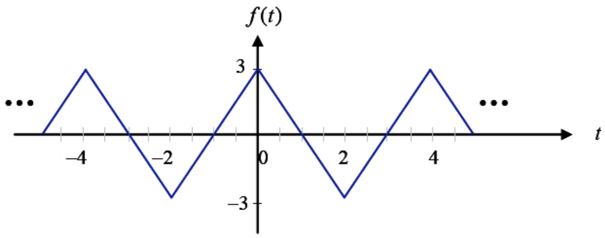
Coefficeints a_n: $a_n^g = a_n^x$ for $n = 1, 2, 3, \ldots$

Coefficeints b_n: $b_n^g = b_n^x$ for $n = 1, 2, 3, \ldots$

15-02-2 10

Full Triangular Signals

Calculate the trigonometric CTFS coefficients of the periodic signal $f(t)$ with fundamental period $T_0 = 4$.



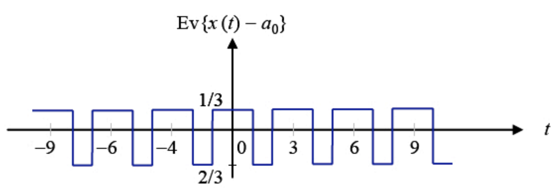
Solution:

$$a_n = \begin{cases} 0 & n \text{ is even} \\ \frac{24}{(n\pi)^2} & n \text{ is odd.} \end{cases}$$

15-02-25 11

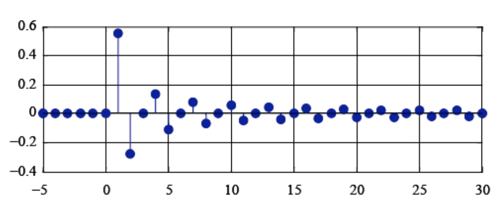
Rectangular Pulse Signals

Calculate the trigonometric CTFS coefficients of the periodic signal $w(t)$ with fundamental period $T_0 = 3$.



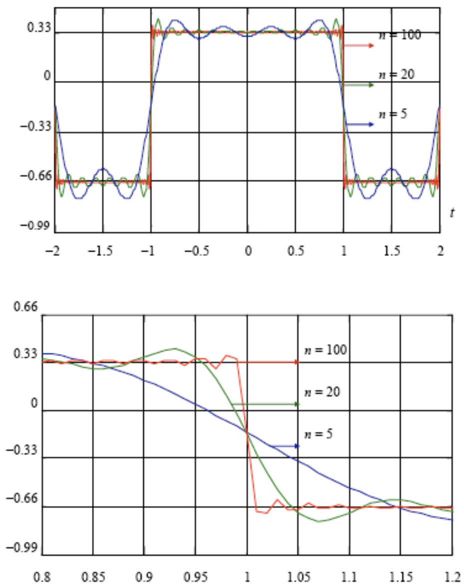
Solution:

$$a_n = \frac{2}{n\pi}\sin\left(\frac{2n\pi}{3}\right)$$



15-02-2 12

Truncated Fourier Series



15-02-25 13

Exponential Fourier Series

An arbitrary periodic function $x(t)$ with fundamental period T_0 can be expressed as

$$x(t) = \sum_{m=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of $x(t)$ and coefficients D_n (the exponential CTFS coefficients) are given by

$$D_n = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-jn\omega_0 t} dt$$

15-02-25 14

Exponential CTFS

- Exponential CTFS is related to trigonometric CTFS:

$$D_n = \begin{cases} a_0 & n = 0 \\ \frac{1}{2}\left(a_n - jb_n\right) & n > 0 \\ \frac{1}{2}\left(a_{-n} + jb_{-n}\right) & n < 0 \end{cases}$$

15-02-25 15

Exponential CTFS

Symmetry Property:

The exponential CTFS coefficients D_n and D_{-n} are complex conjugate of each other for real valued sequences.

Parseval's Theorem:

$$P_x = \frac{1}{T_0} \int_{\langle T_0 \rangle} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |D_n|^2$$

Linearity Property:

$$\text{If } x_1(t) \xleftrightarrow{\text{CTFS}} D_n \quad \text{and } x_2(t) \xleftrightarrow{\text{CTFS}} E_n$$

$$\text{then} \quad a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{\text{CTFS}} a_1 D_n + a_2 E_n$$

Time Shifting Property:

$$\text{If } x(t) \xleftrightarrow{\text{CTFS}} D_n \quad \text{then} \quad x(t - t_0) \xleftrightarrow{\text{CTFS}} D_n \, e^{-jn\omega_0 t_0}$$

Reflection Property:

$$\text{If } x(t) \xleftrightarrow{\text{CTFS}} D_n \quad \text{then} \quad x(-t) \xleftrightarrow{\text{CTFS}} D_{-n} \,,$$

Scaling Property:

$$\text{If } x(t) \xleftrightarrow{\text{CTFS}} D_n \quad \text{then} \quad x(\tfrac{t}{a}) \xleftrightarrow{\text{CTFS}} D_{an} \,,$$

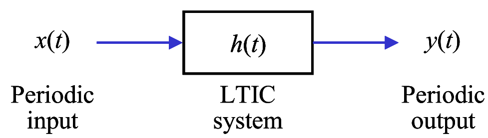
Integration and Differentiation Property:

$$\text{If } x(t) \xleftrightarrow{\text{CTFS}} D_n \quad \text{then}$$

$$\frac{dx}{dt} \xleftrightarrow{\text{CTFS}} jn\omega_0 D_n \quad \text{and} \int_{T_0} x(t)dt \xleftrightarrow{\text{CTFS}} \frac{D_n}{jn\omega_0}.$$

15-02-25 16

LTIC Response to Periodic Signals



- Decomposition in input signals
  - Fourier Spectrum

$$x(t) = \sum_{m=-\infty}^{\infty} D_n e^{jn\omega_0 t},$$

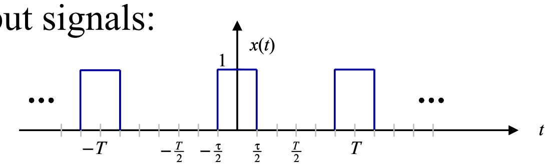
- Superposition in output signals

$$y(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} H(\omega)|_{\omega=n\omega_0}.$$

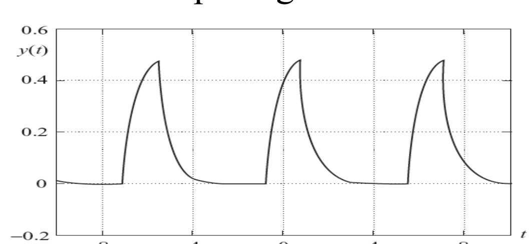
15-02-25 17

LTIC Response to Periodic Signals

- Example
  - Impuse response: $h(t) = \exp(-2t)u(t)$
  - Input signals:



- Superposition in output signals



15-02-25 18

CT Fourier Transform

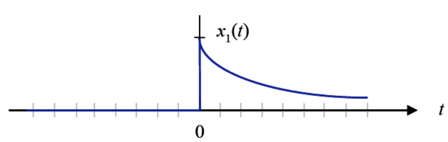
- Valid for both aperiodic and periodic signals
- Derived as a limit of CTFS
- CTFT analysis equation: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
- CTFT synthesis equation: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$
- Existence of CTFT

15-02-25 19

CTFT Example: 1-sided Exponential

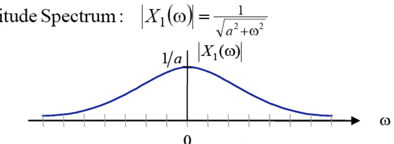
Compute the CTFT of the decaying exponential function

$$x_1(t) = \exp(-at)u(t),\ a \sim R^+.$$

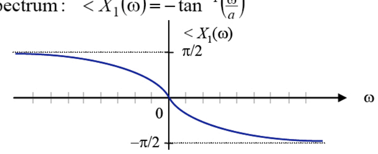


CTFT: $X_1(\omega) = \frac{1}{(a+j\omega)}$

Magnitude Spectrum: $|X_1(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$



Phase Spectrum: $< X_1(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

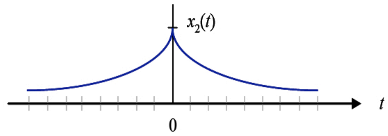


15-02-25 20

CTFT Example: 2-sided Exponential

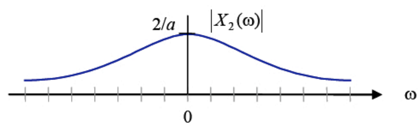
Compute the CTFT of the two sided decaying exponential function

$$x_2(t) = \exp(-|a|t),\ a \sim R^{+}.$$



CTFT : $X_2(\omega) = \frac{2a}{a^2+\omega^2}$

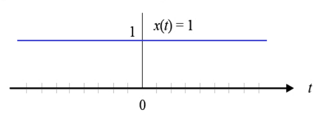
Magnitude Spectrum : $|X_2(\omega)| = \frac{2a}{a^2+\omega^2}$

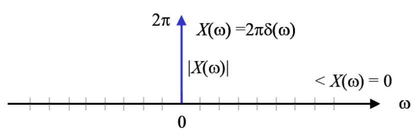


Phase Spectrum : $< X_2(\omega) = 0$

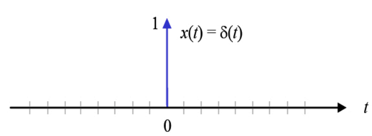
CTFT Example: Constant and impulse

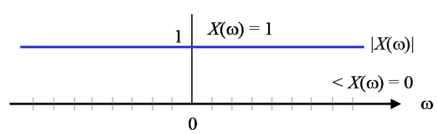
The CTFT of an aperiodic function $g(t)$ is given by $G(\omega) = 2\pi\delta(\omega)$. Determine the aperiodic function $g(t)$.





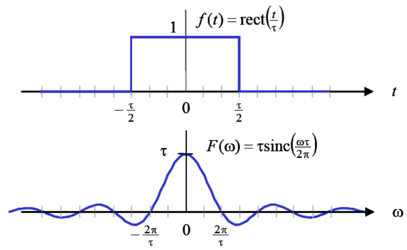
Determine the CTFT of the impulse function $x(t) = \delta(t)$.



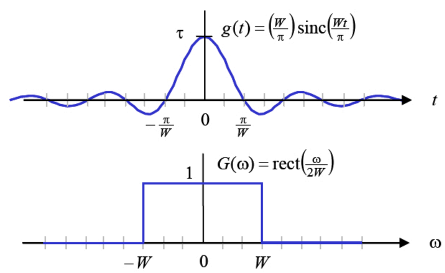


CTFT Example: Gate and Sinc

Compute the CTFT of the gate function.

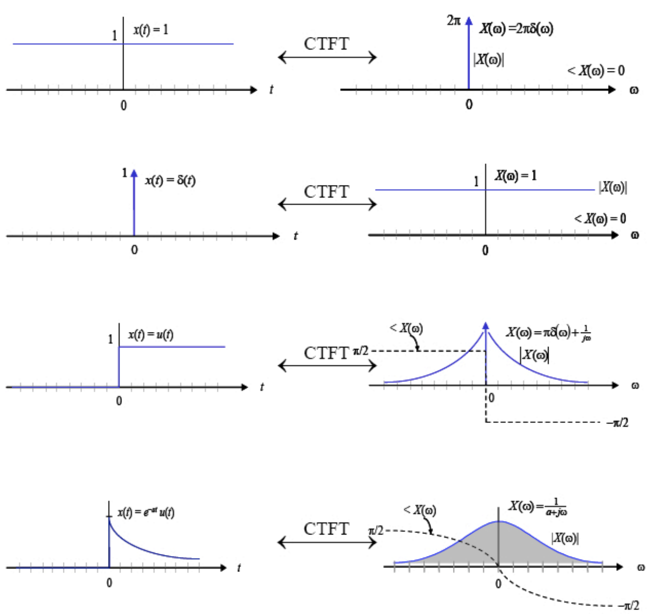


Compute the CTFT of the sinc function.



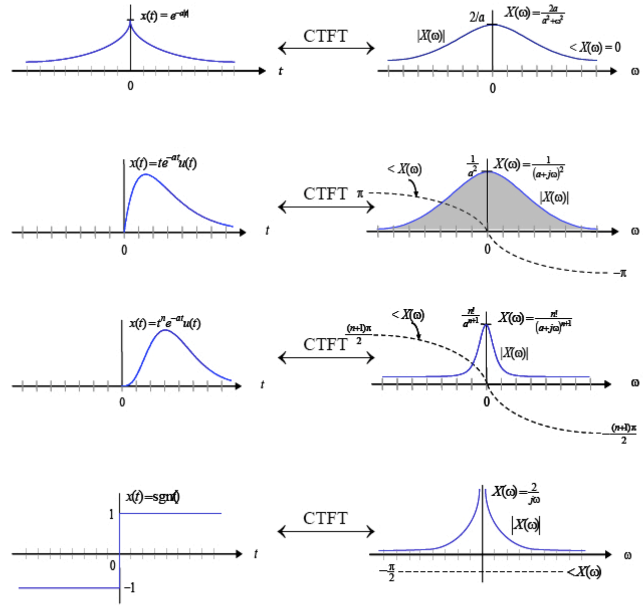
15-02-25 23

CTFT Pairs (I)



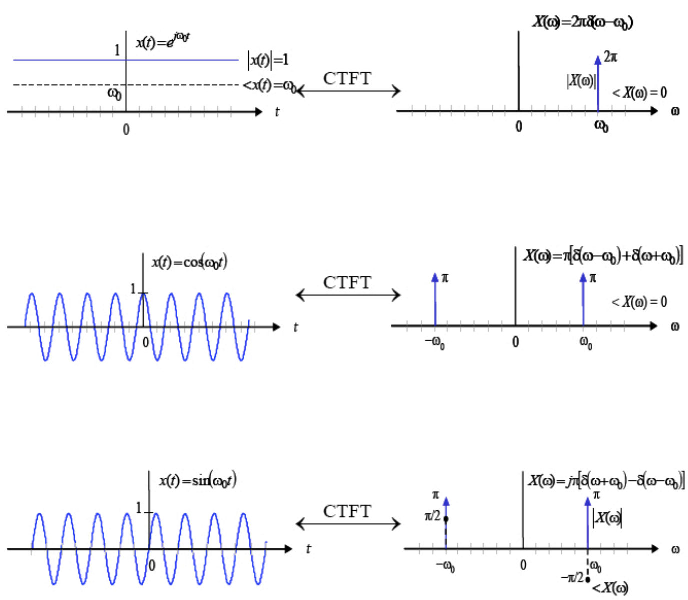
15-02-2 24

CTFT Pairs (II)



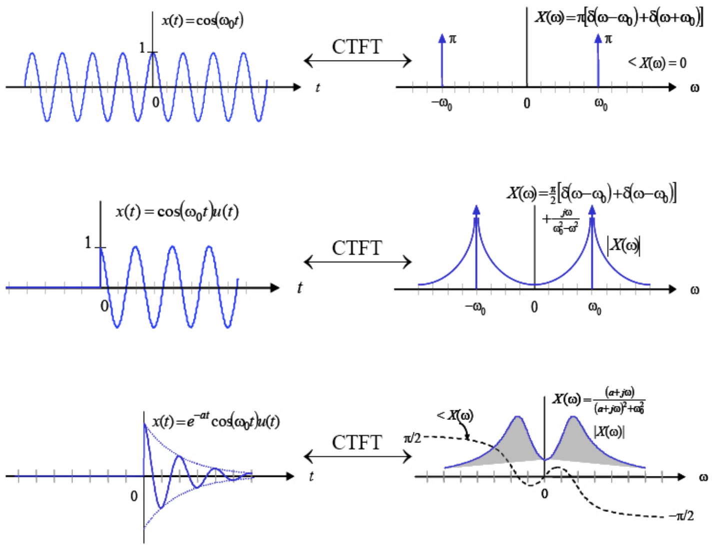
15-02-25 25

CTFT Pairs (III)



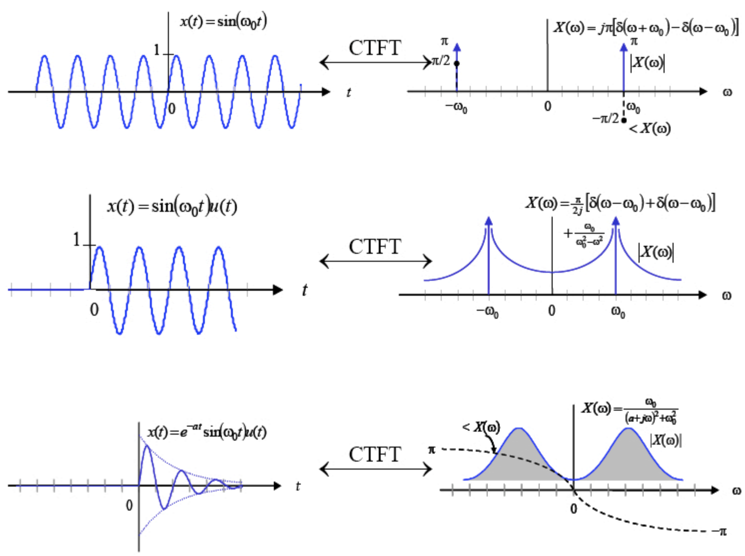
15-02-25 26

CTFT Pairs (IV)



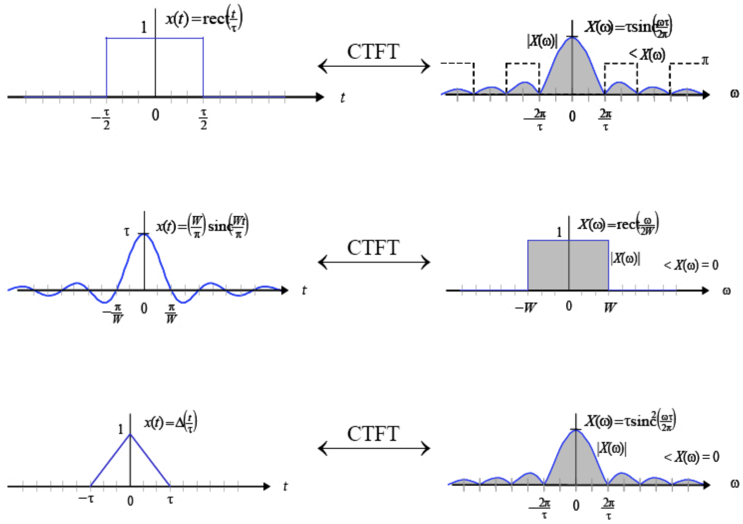
15-02-25 27

CTFT Pairs (V)



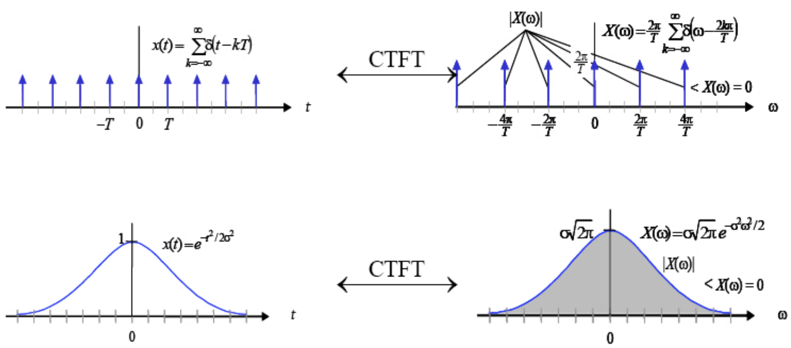
15-02-25 28

CTFT Pairs (VI)



15-02-25 29

CTFT Pairs (VII)



15-02-25 30

Inverse CTFT

- Contour integral using the synthesis equation
- Table look-up
- Partial fraction expansion

CTFT Properties (I)

Transformation properties	Time domain $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	Frequency domain $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Comments
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$	$a_1, a_2 \in C$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$	$a \in \Re$, real-valued
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$	$t_0 \in \Re$, real-valued
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$	$\omega_0 \in \Re$, real-valued
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$	provided dx/dt exists
Time integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$	provided $\int_{-\infty}^{t} x(\tau)d\tau$ exists
Frequency differentiation	$t^n x(t)$	$(j)^n \frac{d^n X}{d\omega^n}$	provided $dX/d\omega$ exists
Duality	$X(t)$	$2\pi x(-\omega)$	if $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$	convolution in time domain
Frequency convolution	$x_1(t) \times x_2(t)$	$\frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$	multiplication in time domain
Parseval's relationship	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$		energy in a signal

CTFT Properties (II)

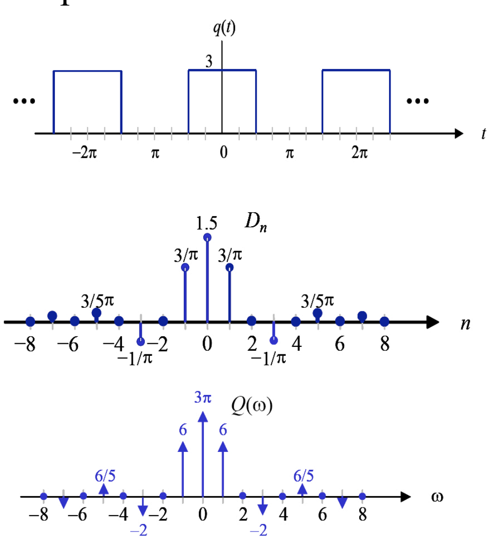
Symmetry properties

Hermitian property	$x(t)$ is a real-valued function	CTFT: $X(-\omega) = X^*(\omega)$ real and imaginary components $\begin{cases} \mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\} \\ \mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\} \end{cases}$ magnitude and phase spectra $\begin{cases} \lvert X(-\omega)\rvert = \lvert X(\omega)\rvert \\ <X(-\omega) = -<X(\omega) \end{cases}$	real component is even; imaginary component is odd magnitude spectrum is even; phase spectrum is odd
Even function	$x(t)$ is even	$X(\omega) = 2\int_0^{\infty} x(t)\cos(\omega t)dt$	simplified CTFT expression for even signals
Odd function	$x(t)$ is odd	$X(\omega) = -j2\int_0^{\infty} x(t)\sin(\omega t)dt$	simplified CTFT expression for odd signals
Real-valued and even function	$x(t)$ is even and real-valued	$\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$ $\mathrm{Im}\{X(\omega)\} = 0$	CTFT is real-valued and even
Real-valued and odd function	$x(t)$ is odd and real-valued	$\mathrm{Re}\{X(\omega)\} = 0$ $\mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\}$	CTFT is imaginary and odd

15-02-25 33

CTFT of Periodic Signals

- CTFS as samples of CTFT



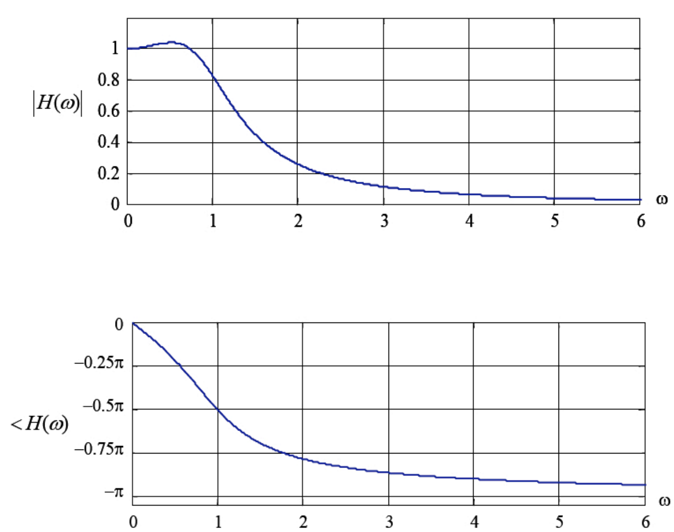
15-02-25 34

Fourier Analysis of LTIC systems

- An LTIC system can be modeled by
  - A Linear constant-coefficient differential equation (with initially rest condition)
  - Impulse Response
  - Frequency Response (CTFT of impulse response)
    - Magnitude (Gain) response
    - Phase Response

15-02-25 35

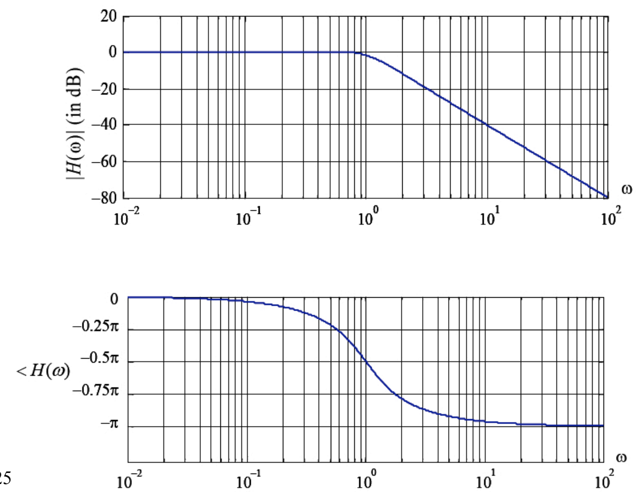
Magnitude and Phase Response



15-02-25 36

Bode Plots

In the Bode plots, the magnitude $|H(\omega)|$ in dB and phase $<H(\omega)$ are plotted as a function of frequency ω using logarithmic scale.



15-02-25 37