EECS2602 Z: Continuous Time Signals and Systems
Instructor: Hui Jiang
Quiz # 3 (10% of the course) your mark: / 50
Time Allowed: 50 minutes

Name: ____

Student ID Number: ____________________ York EECS Email: ____________________

1. [10 points] Assume an input signal $x(t) = A \cdot e^{j\omega_0 t}$ is sent to an LTIC system, whose impulse response is known as $h(t)$, derive the corresponding output signal from the system.

$$y(t) = x(t) \otimes h(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) \cdot A \cdot e^{j\omega_0 (t-\tau)} d\tau$$

$$= A \cdot e^{j\omega_0 t} \int_{-\infty}^{+\infty} h(\tau) \cdot e^{j\omega_0 \tau} d\tau$$

$$= A \cdot e^{j\omega_0 t} \cdot H(\omega_0)$$

where $H(\omega_0) = \int_{-\infty}^{+\infty} h(\tau) \cdot e^{j\omega_0 \tau} d\tau$.

2. [20 points] An input signal $x(t) = e^{-t}u(t)$ is applied to an LTIC system whose impulse response is given by

$$h(t) = \begin{cases} 1 - t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the output of the LTIC system.

Functions $x(\tau)$, $h(\tau)$, $h(-\tau)$, and $h(t-\tau)$are plotted as a function of the variable τ in the top four subplots of Fig. Q1. Depending on the value of t, three cases of convolution may arise.

Case 1: For $t < 0$, the nonzero parts of $h(t-\tau)$ and $x(\tau)$ do not overlap, hence, output $y(t) = 0$.

Case 2: For $0 \le t \le 1$, the nonzero parts of $h(t-\tau)$ and $x(\tau)$ do overlap over the duration $\tau = [0, t]$. Therefore,

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-\tau}(1-t+\tau)d\tau = \underbrace{(1-t)\int_0^t e^{-\tau}\,d\tau}_{\text{Integral I}} + \underbrace{\int_0^t \tau e^{-\tau}\,d\tau}_{\text{Integral II}}.$$

The two integrals simplifies to

$$\text{Integral I} = (1-t)\left[-e^{-\tau}\right]_0 = (1-t)(1-e^{-t}),$$

$$\text{Integral II} = \left[-\tau e^{-\tau} - e^{-\tau}\right]_0 = 1 - e^{-t} - te^{-t}.$$

For $(0 \le t \le 1)$, the output $y(t)$ is given by $y(t) = (1 - t - e^{-t} + te^{-t}) + (1 - e^{-t} - te^{-t}) = (2 - t - 2e^{-t})$.

Case 3: For $t > 1$, the nonzero part of $h(t-\tau)$ completely overlaps with $x(\tau)$ over the region $\tau = [t-1, t]$. Hence,

$$y(t) = \int_{t-1}^{t} x(\tau)h(t-\tau)d\tau = \int_{t-1}^{t} e^{-\tau}(1-t+\tau)d\tau = \underbrace{(1-t)\int_{t-1}^{t} e^{-\tau}\,d\tau}_{\text{Integral I}} + \underbrace{\int_{t-1}^{t} \tau e^{-\tau}\,d\tau}_{\text{Integral II}}.$$

The two integrals simplifies to

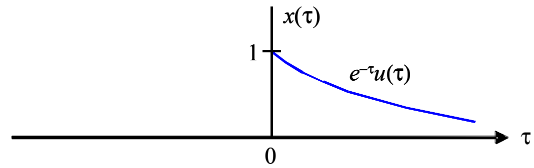
$$\text{Integral I} = (1-t)\left[-e^{-\tau}\right]_{t-1} = (1-t)\left(e^{-(t-1)} - e^{-t}\right),$$

$$\text{Integral II} = \left[-\tau e^{-\tau} - e^{-\tau}\right]_{t-1} = (t-1)e^{-(t-1)} + e^{-(t-1)} - te^{-t} - e^{-t} = te^{-(t-1)} - te^{-t} - e^{-t}.$$

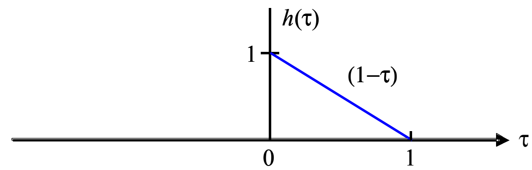
For $(0 \le t \le 1)$, the output $y(t) = \left(e^{-(t-1)} - te^{-(t-1)} - e^{-t} + te^{-t}\right) + \left(te^{-(t-1)} - te^{-t} - e^{-t}\right) = \left(e^{-(t-1)} - 2e^{-t}\right)$.

Combining the above three cases, we obtain $y(t) = \begin{cases} 0 & t < 0 \\ (2 - t - 2e^{-t}) & 0 \le t \le 1 \\ \left(e^{-(t-1)} - 2e^{-t}\right) & t > 1. \end{cases}$

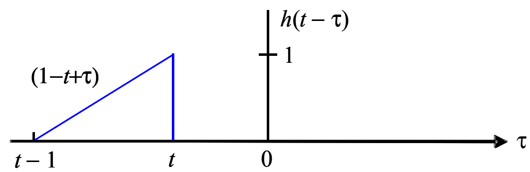
Step 1:



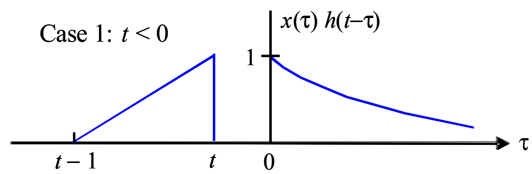
Step 2:



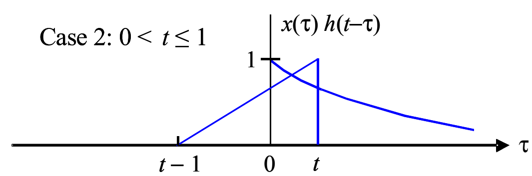
Step 3:



Step 4a:



Step 4b:



Step 4c:

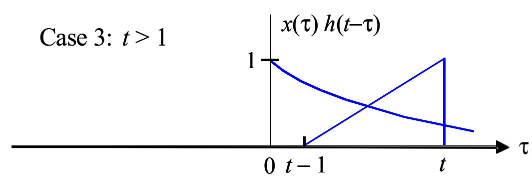
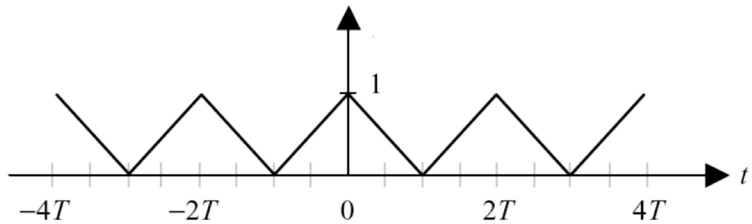


Figure Q1: Convolution of the input signal $x(t)$ with the impulse response $h(t)$.

3. [20 points] For the Sawtooth wave signal (shown below) with period $2T$:

$$x(t) = 1 - \left|\frac{t}{T}\right| \qquad \text{for} \quad -T \leq t < T$$



i) Calculate the trigonometric continuous time Fourier series (CTFS) for the above signal.

ii) Derive the exponential CTFS for this signal.

(d) By inspection, we note that the time period $T_0 = 2T$, which implies that the fundamental frequency $\omega_0 = \pi/T$.

Using Eq. (4.30), the CTFS coefficient T_0 is given by

$$a_0 = \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{T}\int_{0}^{T} x(t)dt = \frac{1}{T} \times \frac{T}{2} = \frac{1}{2}.$$

Using Eq. (4.31), the CTFS cosine coefficients a_n's, for $(n \neq 0)$, are given by

$$a_n = \frac{2}{2T}\int_{-T}^{T} \underbrace{x(t)\cos(n\omega_0 t)}_{\text{even function}} dt = \frac{2}{T}\int_{0}^{T}\left(1-\frac{t}{T}\right)\cos(n\omega_0 t)dt = \frac{2}{T}\int_{0}^{T}\cos(n\omega_0 t)dt - \frac{2}{T^2}\int_{0}^{T} t\cos(n\omega_0 t)dt$$

$$= \frac{2}{n\omega_0 T}\left[\sin(n\omega_0 t)\right]_0^T - \frac{2}{(n\omega_0)^2 T^2}\left[\cos(n\omega_0 t) + n\omega_0 t\sin(n\omega_0 t)\right]_0^T$$

$$= \frac{2}{n\pi}\left[\sin(n\omega_0 T) - 0\right] - \frac{2}{n^2\pi^2}\left[\cos(n\omega_0 T) + n\omega_0 T\sin(n\omega_0 T) - 1\right] \qquad [\because\ \omega_0 T = \pi]$$

$$= \frac{2}{n\pi}\underbrace{\sin(n\pi)}_{=0} - \frac{2}{n^2\pi^2}\left[\cos(n\pi) + n\pi\sin(n\pi) - 1\right]$$

$$= \frac{2}{n^2\pi^2}\left[1-(-1)^n\right] = \begin{cases} 0 & n = even \\ \frac{4}{n^2\pi^2} & n = odd \end{cases}$$

Since $x4(t)$ is even, therefore, the CTFS sine coefficients $b_n = 0$.

$$D_n = \begin{cases} a_0 & n = 0 \\ \frac{1}{2}\left(a_n - jb_n\right) & n > 0 \\ \frac{1}{2}\left(a_{-n} + jb_{-n}\right) & n < 0 \end{cases}$$