COSC 4214: Digital Communications
Instructor: Dr. Amir Asif
Department of Computer Science and Engineering
York University
Handout # 2: Random Signals

Topics:
1. Random Variables: probability density function, mean, variance, and moments
2. Random processes.
3. Ergodic vs. Stationary vs. Wide Sense Stationary Processes
4. Autocorrelation and Power Spectral Density for WSS Processes
5. Additive White Gaussian Noise
6. Signal Transmission through Linear Systems
7. Bandwidth

Sklar: Sections 1.5 – 1.8.

# Random Variables (1)

1. Sample Space: is a set of all possible outcomes

   Example I: $S = \{HH, HT, TH, TT\}$ in tossing of a coin twice.

   Example II: $S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$ in testing three electronic components with $N$ denoting nondefective and $D$ denoting defective.
2. Random variable is a function that associates a real number with each outcome of an experiment.

   Example I: In tossing of a coin, we count the number of heads and call it the RV $X$

   Possible values of $X = 0, 1, 2$.

   Example II: In testing of electronic components, we associate RV $Y$ to the number of defective components. Possible values of $Y = 0, 1, 2, 3$.
3. Discrete RV: takes discrete set of values. RV $X$ and $Y$ in above examples are discrete
4. Continuous RV: takes values on an analog scale.

   Example III: Distance traveled by a car in 5 hours

   Example IV: Measured voltage across a resistor using an analog meter.



# Random Variables (2)

5. Probability density function of a discrete RV: is the distribution of probabilities for different values of the RV.

   Example I: $S = \{HH, HT, TH, TT\}$ in tossing of a coin twice with $X$ = no. of heads

| *Value* ($x$) | 0 | 1 | 2 |
|---|---|---|---|
| $P(X = x)$ | 1/4 | 1/2 | 1/4 |

   Example II: $S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$ in testing electronic components with $Y$ = number of defective components.

| *Value* ($x$) | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | 1/8 | 3/8 | 3/8 | 1/8 |

6. Properties:

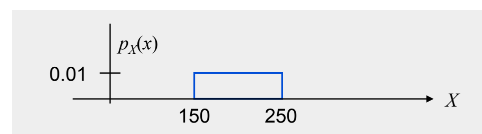
| | | |
|---|---|---|
| a. | $p_X(x) \geq 0$ | always positive |
| b. | $\sum_x p_X(x) = 1$ | adds to 1 |
| c. | $P(X = x) = p_X(x)$ | probability |



# Random Variables (3)

6. Probability density function of a continuous RV: is represented as a continuous function of $X$.

   Example III: Distance traveled by a car in 5 hours has an uniform distribution between 150 and 250 km.




8. Properties of pdf:

| | | |
|---|---|---|
| a. | $p_X(x) \geq 0$ | always positive |
| b. | $\int_{-\infty}^{\infty} p_X(x)dx = 1$ | integrates to 1 |
| c. | $P(a < X < b) = \int_a^b p_X(x)dx$ | probability |

# Random Variables (4)

Activity 1: The pdf of a discrete RV $X$ is given by the following table.

| *Value* $(x)$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | 1/8 | 3/8 | 3/8 | 1/8 |

Calculate the probability $\mathrm{P}(1 \le X < 3)$ and $\mathrm{P}(1 \le X \le 3)$

Activity 2: The pdf of a continuous RV $X$ is $p_X(x) = e^{-x}\, u(x)$. Find the probability $\mathrm{P}(1 < X < 5)$.



# Random Variables (5)

9. Distribution function: is defined as

$$F_X(x) = P(X \le x)$$

which gives

$$F_X(x) = \sum_{x=-\infty}^{x} p_X(x) \qquad \text{for discrete RV}$$

$$F_X(x) = \int_{-\infty}^{x} p_X(x)dx \qquad \text{for continuous RV}$$

9. Moments:

$$E\{X^n\} = \sum_{x=-\infty}^{\infty} x^n p_X(x) \qquad \text{for discrete RV}$$

$$= \int_{-\infty}^{\infty} x^n p_X(x)dx \qquad \text{for continuous RV}$$

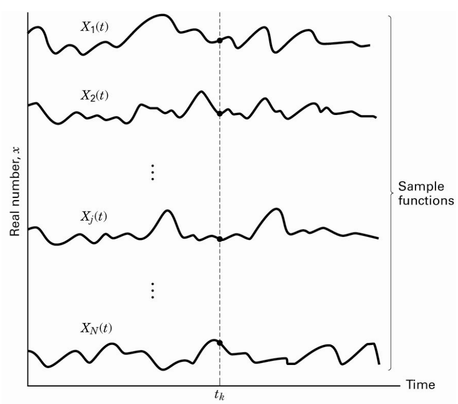
10. Mean is defined as $m_X = E\{X\}$. Variance is defined as $\mathrm{var}\{X\} = E\{X\}^2 - (m_X)^2$.

Activity 3: Calculate and plot the distribution function for pdf's specified in Activities 1 and 2. Also calculate the mean and variance in each case.



# Random Processes (1)

1. The outcome of a random process is a time varying function. Examples of ransom processes are: temperature of a room; output of an amplifier; or luminance of a bulb.




2. A random process can also be thought of as a collection of RV's for specified time instants. For example, $X(t_k)$, measured at $t = t_k$ is a RV.



# Random Processes (2)

3. Random processes are often specified by their mean and autocorrelation.
4. Mean is defined as

$$E\{X(t_k)\} = \sum_{x=-\infty}^{x} X(t_k)\, p_{X_k}(x) \qquad \text{for discrete-time random process}$$

$$E\{X(t_k)\} = \int_{-\infty}^{x} X(t_k)\, p_{X_k}(x)dx \qquad \text{for continuous-time random process}$$

5. Autocorrelation is defined as

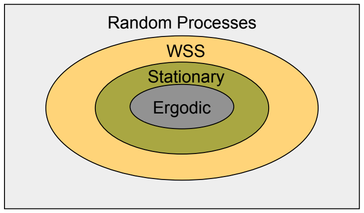
$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\}$$

Activity 4: Consider a random process

$$X(t) = A\cos(2\pi f_0 t + \phi)$$

where $A$ and $f_0$ are constants, while $\phi$ is a uniformly distributed RV over $(0, 2\pi)$. Calculate the mean and autocorrelation for the aforementioned process.

# Classification of Random Processes (1)




1. Wide Sense Stationary (WSS) Process: A random process is said to be WSS if its mean and autocorrelation is not affected with a shift in the time origin

$$E\{x(t)\} = m_X = \text{constant} \quad \text{and} \quad R_X(t_1,t_2) = R_X(t_1 - t_2)$$

2. Strict Sense Stationary (SSS) Process: A random process is said to be SSS if none of its statistics change with a shift in the time origin

$$p_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k;t_1,t_2,\ldots,t_k) = p_{X_1,X_2,\ldots,X_k}(x_1,x_2,\ldots,x_k;t_1+T,t_2+T,\ldots,t_k+T)$$

3. Ergodic Process: Time averages equal the statistical averages.



# Classification of Random Processes (2)

Activity 5: Show that the random process in Activity 4 is a WSS process.

4. For WSS processes, the autocorrelation can be expressed as a function of single variable

$$R_X(t_1,t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

5. Autocorrelation satisfies the following properties

| | | |
|---|---|---|
| 1. | $R_x(\tau) = R_x(-\tau)$ | Even function w. r. t. $\tau$ |
| 2. | $R_x(\tau) \le R_x(0)$ | Maximum occurs at $\tau = 0$ |
| 3. | $R_x(\tau) \xleftrightarrow{FT} G_x(f)$ | Fourier transform pairs |
| 4. | $R_x(0) = E\{X^2(t)\}$ | Correlation |

6. Fourier transform of autocorrelation is referred to as the power spectral density (PSD)

| | | |
|---|---|---|
| 1. | $G_x(f) \ge 0$ | Always real valued |
| 2. | $G_x(f) = G_x(-f)$ | Even function |
| 3. | $R_x(\tau) \xleftrightarrow{FT} G_x(f)$ | Fourier transform pairs |
| 4. | $P_X = \int_{-\infty}^{\infty} G_x(f)\,df$ | Variance |



# Classification of Random Processes (3)

Activity 6: Determine which of the following are valid autocorrelation function

$$1.\quad R_x(\tau) = \begin{cases} 1 & -1 \le \tau \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$2.\quad R_x(\tau) = \delta(\tau) + \sin(2\pi f_0 t)$$

$$3.\quad R_x(\tau) = \exp(|\tau|)$$

Activity 7: Determine which of the following are valid power spectral density function

$$1.\quad S_x(f) = 10 + \delta(f - 10)$$

$$2.\quad S_x(f) = \delta(\tau) + \cos^2(2\pi f_0 t)$$

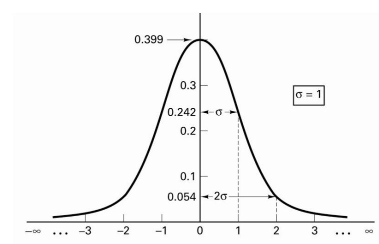
$$3.\quad S_x(f) = \exp(-2\pi(f^2 - 10))$$



# Additive Gaussian Noise

1. Noise refers to unwanted interference that tends to obscure the information bearing signal
2. Noise can be classified into two categories:
   a) Man-made Noise introduced by switching transients and simultaneous presence of neighboring signals
   b) Natural Noise produced by the atmosphere, galactic sources, and heating up of electrical components. The latter is referred to as the thermal noise.
3. Thermal noise is difficult to be eliminated and often modeled by the Gaussian probability density function

$$p_N(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$




which has a mean $\mu_n = 0$ and $\text{var}(n) = \sigma^2$.

# Additive White Gaussian Noise

4. Additive Gaussian Noise: refers to the following model for introduction of noise in the signal

$$z = a + n \quad \text{random variable}$$
$$z(t) = A + n(t) \quad \text{random process}$$

5. Given that the noise $n$ is a Gaussian RV and $a$ is the dc component, which is constant, the pdf of $z$ is given by

$$p_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

which has a mean $\mu_n = a$ and var($n$) = $\sigma^2$.

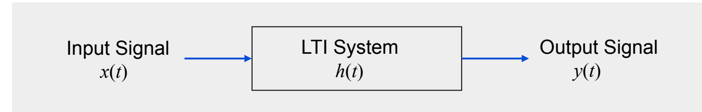
6. Additive White Gaussian Noise (AWGN): adds an additional constraint on the power spectral density

$$R_n(\tau) = \frac{N_0}{2}\delta(\tau) \xleftrightarrow{FT} G_n(f) = \frac{N_0}{2}$$

Activity 8: Calculate the variance of AWGN given its PSD is $N_0/2$.



# Signal Processing with Linear Systems (1)




1. For deterministic signals, the output of the LTI system is given by
   a) Convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\alpha)h(t-\alpha)d\alpha$$

   b) Transfer function:

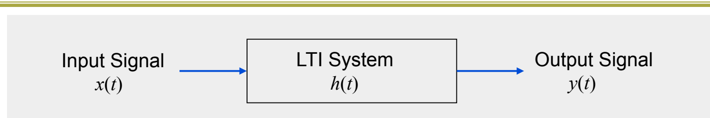
$$y(t) = \Im^{-1}[X(f)H(f)]$$

   where $X(f)$ and $H(f)$ are Fourier transforms of $x(t)$ and $h(t)$.

Activity 9: Determine the output of the LTI system if the input signal $x(t) = e^{-at}\, u(t)$ and the transfer function $h(t) = e^{-bt}\, u(t)$ with $a \neq b$.



# Signal Processing with Linear Systems (2)




2. For WSS random processes, statistics of the output of the LTI system can only be evaluated using the following formula.

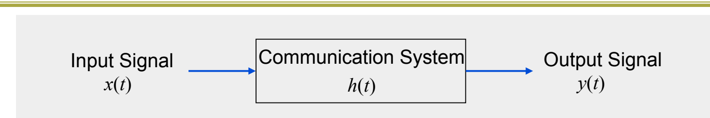
$$\text{Mean}: \quad \mu_y = \mu_x \int_{-\infty}^{\infty} h(t)dt$$
$$\text{Autocorrelation}: \quad R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau)$$
$$\text{PSD}: \quad S_y(f) = S_x(f)\,|H(f)|^2$$

Activity 10: Derive the above expressions for WSS random processes.

Activity 11: Calculate the mean and autocorrelation of the output of the LTI system if the input $x(t)$ to the system is White Noise with PSD of $N_0/2$ and the impulse response of the system is given by $h(t) = e^{-bt}\, u(t)$.



# Distortionless Transmission




1. For distortionless transmission, the signal can only undergo
   a) Amplification or attenuation by a constant factor of $K$
   b) Time delay of $t_0$

   In other words, there is no change in the shape of the signal
2. For distortionless transmission, the received signal must be given by

$$y(t) = Kx(t - t_0)$$

3. Based on the above model, the transfer function of the overall communication system is given by

$$H(f) = Ke^{-j2\pi f t_0}$$

   with impulse response

$$h(t) = K\delta(t - t_0).$$

# Ideal Filters

Lowpass Filter:

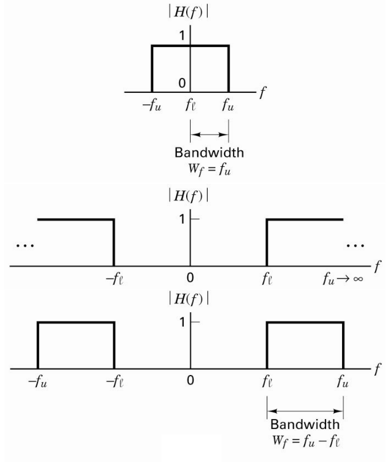
$$H(f)=\begin{cases} e^{-j2\pi f t_0} & |f|<f_u \\ 0 & |f|\ge f_u \end{cases}$$

Highpass Filter:

$$H(f)=\begin{cases} 0 & |f|<f_\ell \\ e^{-j2\pi f t_0} & |f|\ge f_\ell \end{cases}$$

Bandpass Filter:

$$H(f)=\begin{cases} 0 & |f|\le f_\ell \\ e^{-j2\pi f t_0} & f_\ell<|f|<f_u \\ 0 & |f|\ge f_u \end{cases}$$



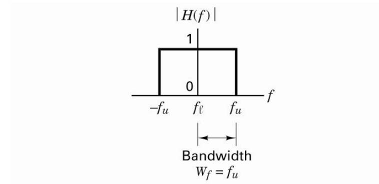

Activity 12: Calculate the impulse response for each of the three ideal filters.

Activity 13: Calculate the PSD and autocorrelation of the output of the LPF if WGN with PSD of $N_0/2$ is applied at the input of the LPF.

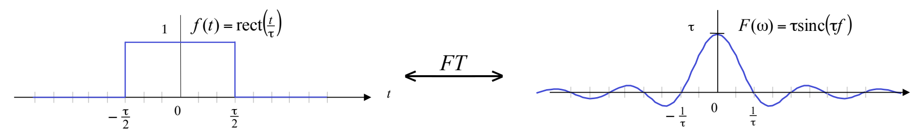


# Bandwidth

1. For baseband signals, absolute bandwidth is defined as the difference between the maximum and minimum frequency present in a signal.

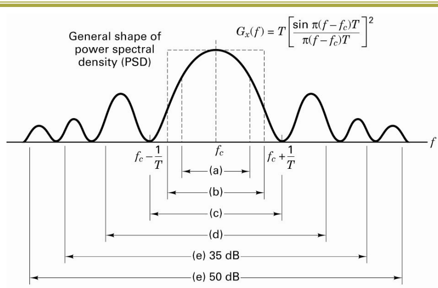



2. Most time limited signals are not band limited so strictly speaking, their absolute bandwidth approaches infinity






# Bandwidth for Bandpass signals(2)




Alternate definitions of bandwidth include:

(a) Half-power Bandwidth: Interval between frequencies where PSD drops to 0.707 (3dB) of the peak value.

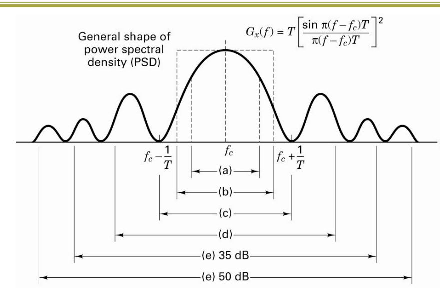
(b) Noise Equivalent Bandwidth is the ratio of the total signal power ($P_x$) over all frequencies to the maximum value of PSD $G_x(f_c)$.

(c) Null to Null Bandwidth: is the width of the main spectral lobe.

(d) Fractional Power Containment Bandwidth: is the frequency band centered around $f_c$ containing 99% of the signal power



# Bandwidth for Bandpass signals(3)




Alternate definitions of bandwidth include:

(e) Bounded Power Spectral Density: the width of the band outside which the PSD has dropped to a certain specified level (35dB, 50dB) od the peak value.

(f) Absolute Bandwidth: Band outside which the PSD = 0.