L10 Matched Filtering & Optimum Detection

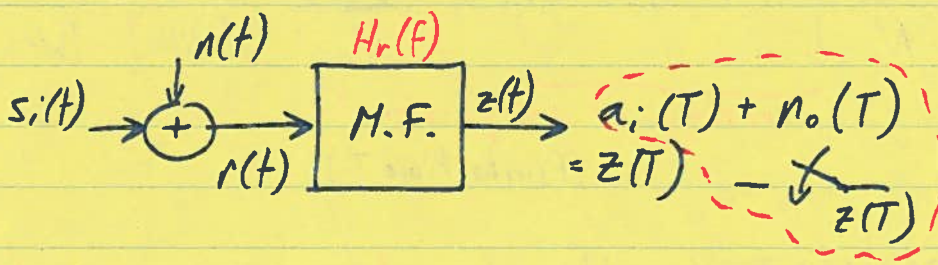
10.1 Matched Filter

- For ML knowing $P_B = Q\left(\frac{a_1 - a_2}{2\sigma_0}\right)$ we are looking to maximize $\frac{a_1 - a_2}{2\sigma_0}$ — $\max(a_1 - a_2)$ is obvious

but $\max\left(\frac{a_i}{\sigma_0}\right)$ is also important, which is prop. to $\max\left(\frac{a_i^2}{\sigma_0^2}\right) = \left(\frac{S}{N}\right)_T$

signal power at time T to the avg. noise power at time T

i.e. considering Finite signal over time T

- 

- build some filter that maximizes SNR_T
- signal through filter: $A(f) = H_r(f)\,S(f)$

$$a(t) = F.T.^{-1}\{A(f)\} = \int_{-\infty}^{\infty} H_r(f)\,S(f)\,e^{j2\pi f t}\,df$$

- noise through filter (PSD)

$$N_0(f) = \frac{N_0}{2}\,|H_r(f)|^2$$

$$\sigma_0^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |H_r(f)|^2\,df$$

$$\boxed{SNR_T = \frac{\left|\int_{-\infty}^{\infty} H_r(f)\,S(f)\,e^{j2\pi f T}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H_r(f)|^2\,df}}$$

- to keep going invoke Schwarz Inequality

$$\left|\int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df\right|^2 \le \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df$$

$$\therefore\ SNR_T \le \frac{\int_{-\infty}^{\infty} |H_r(f)|^2 df \int_{-\infty}^{\infty} \left|S^*(f)\,e^{-j2\pi fT}\right|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H_r(f)|^2 df}$$

$$SNR_T \le \frac{2}{N_0}\underbrace{\int_{-\infty}^{\infty} |S(f)|^2 df}_{\text{energy of energy signal}} = \frac{2E_b}{N_0} \quad \frac{[J]}{[W/Hz]} = \frac{[J]}{[W\cdot s]} = \frac{[J]}{[J]}$$

(finite time T)

Schwarz EQUALITY holds if (i.e. max. SNR_T)

$$X(f) = k\,Y(f)$$

$x(t-t_0) \rightarrow X(f)e^{-j2\pi}$

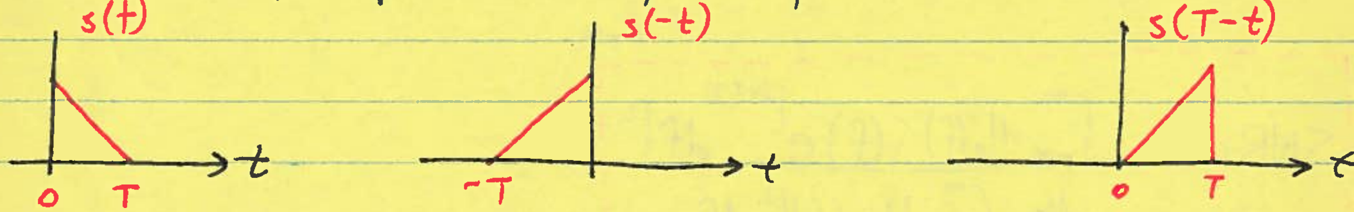
$X^*(f) = X(-f)$

$x(-t) \Leftrightarrow X(-f)$

$$\therefore \text{ setting } \quad H_r(f) = k\,S^*(f)\,e^{-j2\pi fT} = k\,S(-f)\,e^{j2\pi fT}$$

$$h_r(t) = F.T.^{-1}\{H_r(f)\} = \boxed{k\,s(T-t) = h_r(t)}$$

$\therefore$ receive (matched) filter impulse response is mirror image of $s(t)$ delayed by T



10.2 Correlator Equivalent

- Another way to view the M.F.

$$r(t) \longrightarrow \boxed{h_r(t)} \longrightarrow z(t)$$

$$z(t) = \int_{-\infty}^{\infty} r(t-\tau)h_r(\tau)\,d\tau$$

- for causal system — if i/p starts at $t=0$

$$z(t) = \int_{0}^{\infty} r(t-\tau)h_r(\tau)\,d\tau \quad \left(z(t) = \int_0^t r(t-\tau)h(\tau)\,d\tau \right)$$

- w/i change of variables

$$\left.\begin{array}{ll} s = t-\tau & \tau = t-s \\ ds = -d\tau & \tau|_0 = s|_t \\ & \tau|^{\infty} = s|^{-\infty} \end{array}\right\} \quad \begin{aligned} z(t) &= \int_t^{-\infty} r(s)h(t-s)(-ds) \\ &= \int_{-\infty}^{t} r(s)h(t-s)\,ds \end{aligned}$$

$$z(t) = \int_{-\infty}^{t} r(\tau)h_r(t-\tau)\,d\tau \quad \left(z(t) = \int_0^t r(\tau)h(t-\tau)\,d\tau \right)$$

- substitute MF impulse response

$$z(t) = \int_0^t r(\tau)\,s(T-(t-\tau))\,d\tau$$

$$z(t) = \int_0^t r(\tau)\,s(T-t+\tau)\,d\tau$$

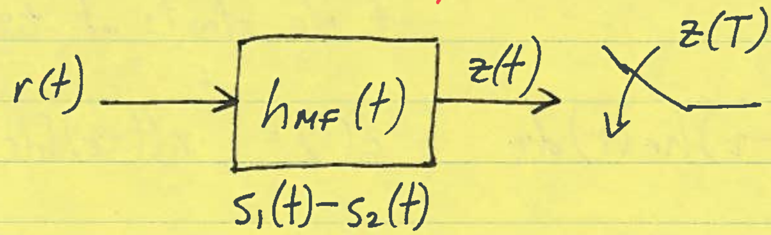
at $t = T\ldots$ $\longrightarrow$

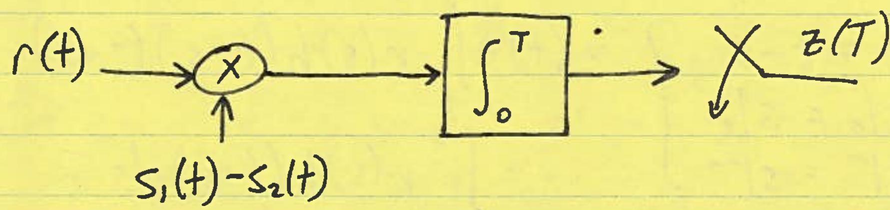
$$z(T) = \int_0^T r(\tau)\, s(\tau)\, d\tau \qquad \text{recall } R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

a cross-correlation $\qquad R_x(0) = \int_{-\infty}^{\infty} x(t)\, x(t)\, dt$

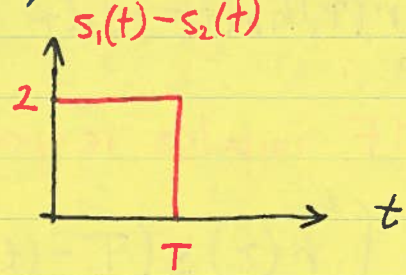
10.3 Receive Filter Arrangements

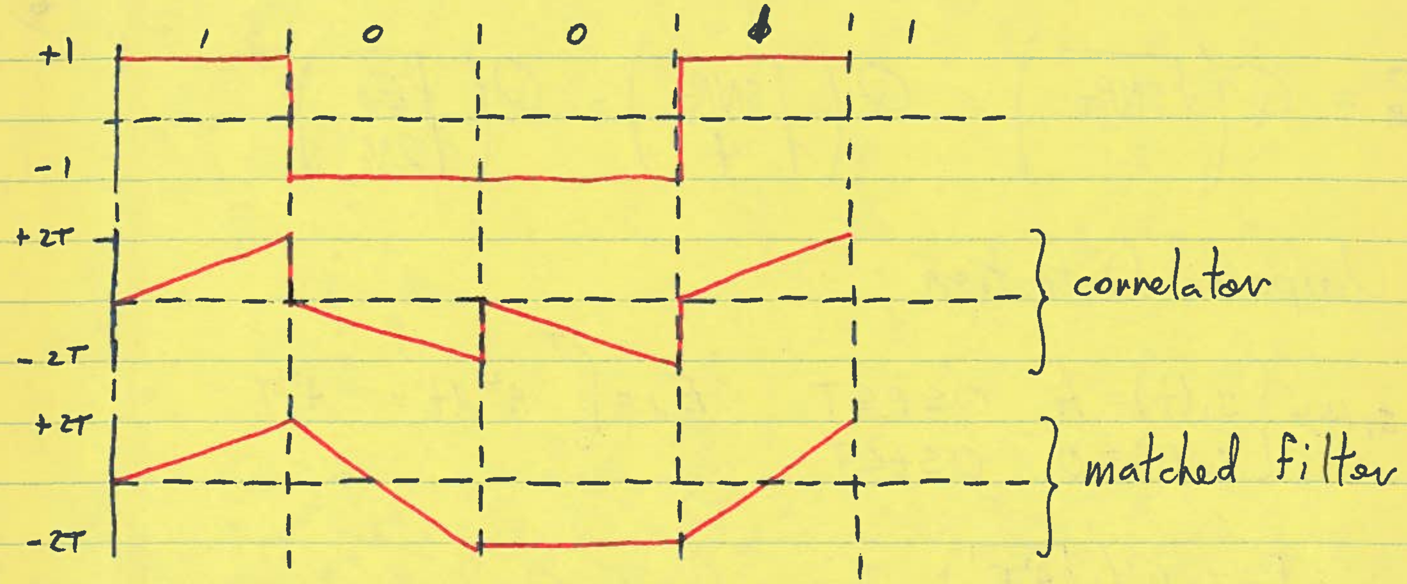
- looking to max. $\dfrac{a_1 - a_2}{\sigma_0}$
- so match to the signal difference





- MF and correlator only match results at $t = T$! ! !
- e.g. +1/-1 NRZ





- correlator is a little easier to make (multiplier-accumulator units) and can be a little more forgiving of timing errors in bandpass systems

- matched filter is easier to realize as part of a more complex system that merges a number of filtering functions (e.g. whitening + equalization)

10.4 Optimum Detection

- ML-D combined with matched filter/correlator results in optimum detection

inst. pwr. @T

to minimize $P_B = Q\left(\dfrac{a_1 - a_2}{2\sigma_0}\right)$ we maximize $SNR_T = \dfrac{(a_1-a_2)^2}{\sigma_0^2}$ ← avg. pw

ratio of avg. power

- through MF/corr. SNR_T = we showed this power ratio is equivalent to the energy ratio

energy of diff signal $E_d = \int_0^T [s_1(t) - s_2(t)]^2 d$

$$SNR_T = \frac{(a_1 - a_2)^2}{\sigma_0^2} = \frac{E_d}{N_0/2}$$

$$P_B = Q\left(\frac{\sqrt{SNR_T}}{2}\right) = Q\left(\sqrt{\frac{SNR_T}{4}}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

10.5 Unipolar Detection

$$s_i(t) = \begin{cases} s_1(t) = A & 0 \le t \le T \\ s_2(t) = 0 & 0 \le t \le T \end{cases} \qquad E_d = \int_0^T A^2\,dt = A^2 T$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{A^2T}{2N_0}}\right)$$

- in terms of avg. bit energy?

$$E_b = \frac{1}{2}E_1 + \frac{1}{2}E_2 = \frac{1}{2}\int_0^T A^2\,dt + \frac{1}{2}\int_0^T 0\,dt = \frac{A^2T}{2}$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

10.6 Bipolar Detection

$$s_i(t) = \begin{cases} s_1(t) = +A \\ s_2(t) = -A \end{cases} \quad E_d = \int_0^T 4A^2\,dt = 4A^2T$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

$$E_b = \frac{1}{2}A^2T + \frac{1}{2}A^2T = A^2T$$

$$\therefore \quad P_B = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \leftarrow \text{which is better}$$

10.7 Waterfall Curves