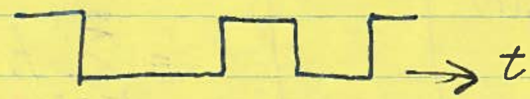
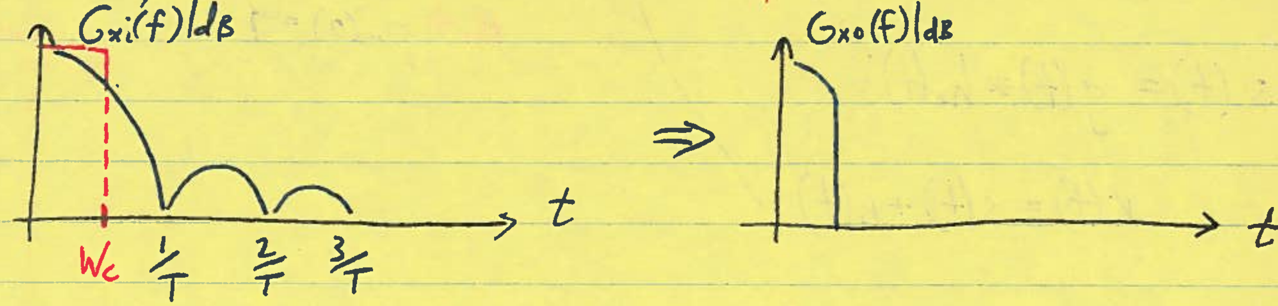
11 Intersymbol Interference

11.1 The Problem

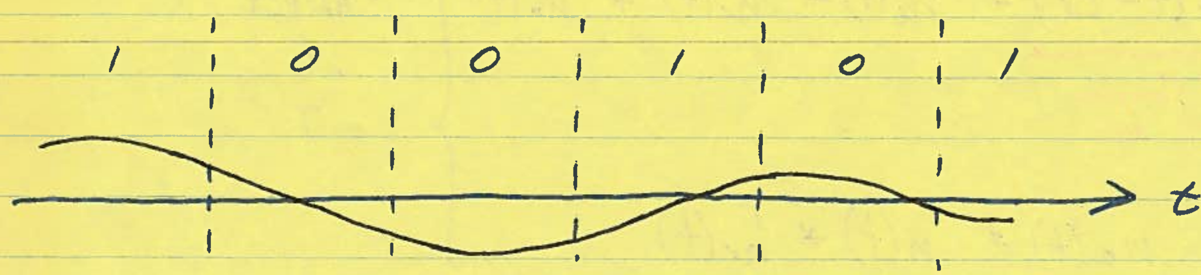
- It would be nice if we could just send simple pulses like NRZ



- But you have limited channel spectrum

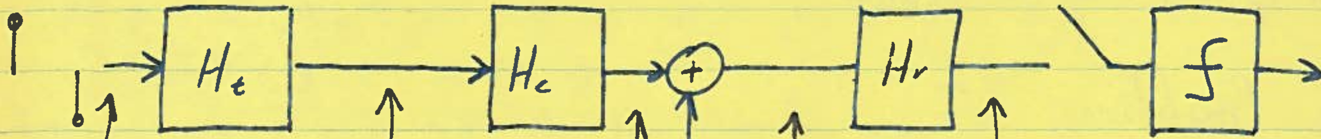


- thus experience distortion



11.2 A System Model

- what we are dealing with more formally



$n(t)$

$\sum_{k=-\infty}^{+\infty} a_k \delta(t-kT)$

$$g(t) = \sum_{k=-\infty}^{+\infty} a_k \delta(t-kT) * h_t(t)$$

$$= \sum_{k=-\infty}^{+\infty} a_k h_t(t-kT)$$

$$s(t) = g(t) * h_c(t)$$

$$r(t) = s(t) + n(t)$$

$$z(t) = r(t) * h_r(t)$$

$$= \sum_{k=-\infty}^{\infty} A a_k p_r(t - \cancel{t_d} - kT) + n_o(t)$$

choose A such that $p_r(0) = 1$

for simplicity ignore t_d

in summary:

delay from TX i/p to RX o/p

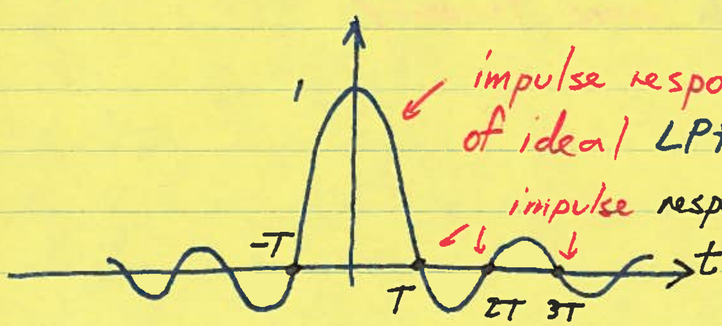
$$A \underbrace{p_r(t - \cancel{t_d})}_{\text{pulse shape at RX o/p}} = h_t(t) * h_c(t) * h_r(t)$$

$$n_o(t) = n(t) * h_r(t)$$

SAMPLE at mT

- to pick out (sample) only one symbol at a time an ideal respone would o/p

$$p_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(t/T)$$



impulse response of ideal LPF

impulse response $= 0$ @ mT $\quad m > 0$

$$z(mT) = A a_m p_r(0) + \underbrace{\sum_{\substack{k=-\infty \\ k \neq m}}^{\infty} A a_k p_r[(m-k)T]}_{\text{ISI term}} + N_m$$

$m = 0, \pm 1, \pm 2, \ldots$

ISI term ... make this $\emptyset$!!!

$$N_m = n_o(mT) = [n(t) * h_r(t)]\Big|_{t=mT}$$

11.3 Nyquist Pulses

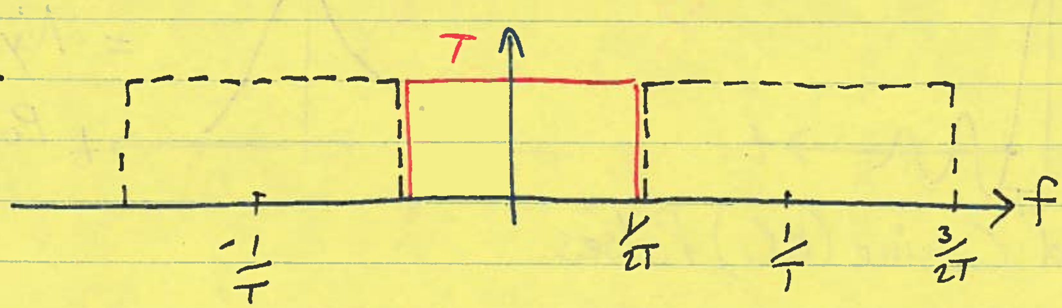
want $\quad p_r(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ $\quad$ recall: $p_r = h_t(t) * h_c(t) * h_r(t)$

- to achieve this employ Nyquist Pulse Shaping Criterion

$$P_r(f) = F.T.\{p_r(t)\}$$

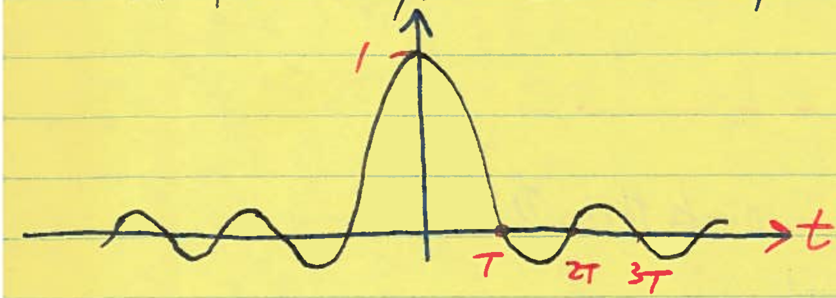
$$\sum_{k=-\infty}^{\infty} P_r\left(f+\frac{k}{T}\right) = T$$

e.g.



in this case

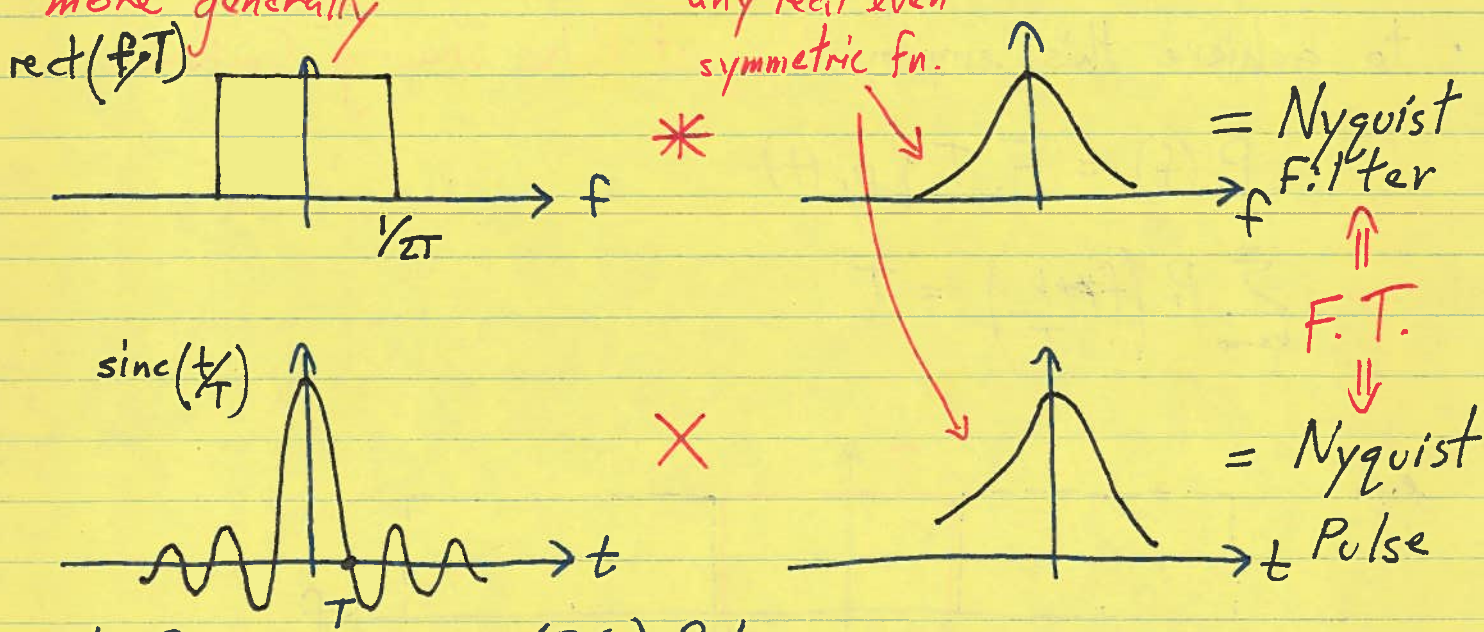
- in this "simple" case $p_r(t) = \dfrac{\sin(\pi t/T)}{\pi t/T} = \text{sinc}(t/T)$



- intuitively obvious ...
  - receiver "samples" at T impulses sent every T
  - "anti-aliasing" needs $f_0 > 1/2T$
  - impulse response of such filter is $\text{sinc}(t/T)$

IDEAL NYQUIST PULSE from IDEAL NYQUIST FILTER

- more generally



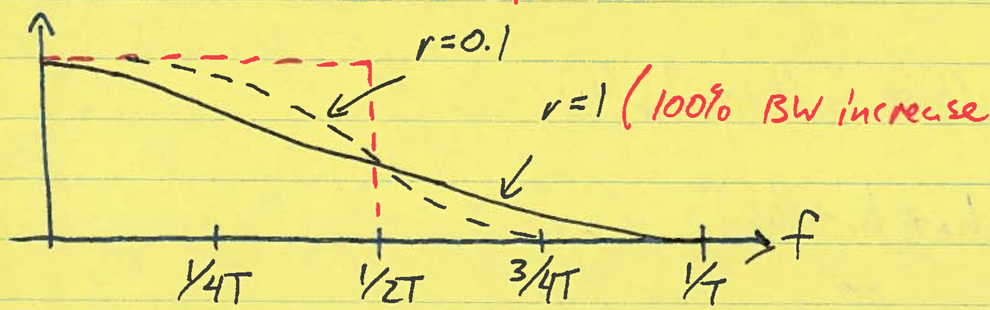
11.4 Raised Cosine (RC) Pulses

$$P_{RC}(f) = \begin{cases} T & 0 \le |f| \le (1-r)/2T \\ \dfrac{T}{2}\left\{1 + \cos\left[\dfrac{\pi T}{r}\left(|f| - \dfrac{(1-r)}{2T}\right)\right]\right\} & (1-r)/2T \le |f| \le (1+r)/2T \\ 0 & \underbrace{(1+r)/2T}_{W_{RC}} < |f| \end{cases}$$

$W_0 = \frac{1}{2T}$ $\quad W_{RC} = W_0 + W_0 \cdot r$

$$r = \frac{W_{RC} - W_0}{W_0} = \frac{W_{RC}}{W_0} - 1$$

roll-off factor



$$p_{RC}(t) = \text{sinc}\left(\frac{t}{T}\right) \cdot \frac{\cos(\pi r t/T)}{1-(2rt/T)^2}$$

11.5 Optimum TX & RX Filters

Again...

$$z(mT) = A \cdot a_m \cdot p_r(0) + N_m = A \cdot a_m + N_m$$

IFF we achieve ZERO ISI

- in which case prob. of error is...

$$P_b = P\left(z(mT) > 0 \mid a_m = -1\right)$$

OK iff $P(a_m = 1) = P(a_m = -1) = 1/2$

$$= P_n\left(N_m > A\right) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_A^\infty e^{-u^2/2\sigma_n^2}\, du$$

$$= Q\left(\frac{A}{\sigma_n}\right)$$ ← Minimize this!!!

minimize $\dfrac{\sigma_n^2}{A^2}$!!!

$$var(N_n) = \sigma_n^2 = \int_{-\infty}^{\infty} G_n(f)\,|H_r(f)|^2\,df$$

$$A\,p_r(t) = h_t(t) * h_c(t) * h_r(t)$$

$$A^2 p_r^2(t) = (h_t * h_c * h_r)^2$$

$$A^2 \int_{-\infty}^{\infty} p_r^2(t)\,dt = \int_{-\infty}^{\infty} (h_t * h_c * h_r)^2\,dt$$

Parseval's thm.

$$A^2 = \int_{-\infty}^{\infty} \frac{|H_t|^2 |H_c|^2 |H_r|^2}{|P_r|^2}\,df$$

$$\frac{\sigma_n^2}{A^2} = \int_{-\infty}^{\infty} G_n(f)\,|H_r|^2\,df \cdot \int_{-\infty}^{\infty} \frac{|P_r|^2}{|H_t|^2 |H_c|^2 |H_r|^2}\,df \geq \left[\int_{-\infty}^{\infty} \frac{G_n^{1/2}\,|P_r|}{|H_t|\,|H_c|}\,df\right]^2$$

again using Schwarz's inequality $\left|\int X Y^* df\right|^2 \leq \int |X|^2 df \int |Y|^2 df$

equality if $|X| = \alpha |Y|$

$$|H_r(f)| = \frac{\alpha\,|P_r|^{1/2}}{G_n^{1/4}\,|H_c|^{1/2}} \qquad |H_t(f)| = \frac{(A/\alpha)\,|P_r|^{1/2}\,G_n^{1/4}}{|H_c|^{1/2}}$$

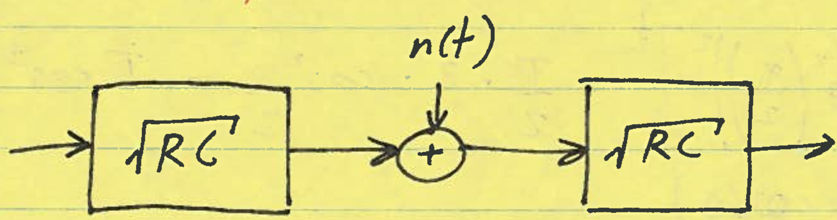
can show... $P_{b,min} = Q\left\{\sqrt{E_T}\left[\int_{-\infty}^{\infty} \frac{G_n^{1/2}\,|P_r|}{|H_c|}\right]^{-1}\right\}$

where $E_T = \int_{-\infty}^{\infty} |H_t|^2 df$ ← transmitted energy per symbol

note for $H_c = 1$ and $G_n = \frac{N_0}{2}$

$|H_r| = \eta |P_r|^{\frac{1}{2}}$ $\qquad |H_t| = \eta_2 |P_r|^{\frac{1}{2}}$

arbitrary constant



get M.F. & ISI simultaneously

11.6 Non-Ideal Channel Example

- binary comms. system $R = 9600$ bps
- $H_c = \dfrac{1}{1 + j\frac{f}{4800}}$, • $\frac{N_0}{2} = 10^{-12} \ \frac{W}{Hz}$
- want to receive RC waveform with $r = 1$ and $T = \frac{1}{9600}$ (obviously) } zero ISI
- find H_t & H_r for optimum detection
- find TX signal energy to get $P_{b,min} = 10^{-6}$

want

$$P_{RC}(f) = \frac{T}{2}\Big(1 + \cos\big(\pi T f\big)\Big) \qquad 0 \le f < \frac{1}{T} = 9600$$

$$\therefore \; H_r(f) = \frac{\alpha}{G_n^{1/4}} \frac{|P_{RC}|^{1/2}}{|H_c|^{1/2}} \qquad H_t(f) = \frac{A}{\alpha} G_n^{1/4} \cdot \frac{1}{|H_c|^{1/2}} \cdot |P_r|^{1/2}$$

$$|P_{RC}| = \left[\frac{T^2}{4}\big(1 + \cos(\pi T f)\big)^2\right]^{1/2} = \left[\frac{T^2}{4}\big(1 + \cos x\big)^2\right]^{1/2}$$

$$= \left[\frac{T^2}{4}\left(2\cos^2\left(\frac{x}{2}\right)\right)^2\right]^{1/2} = \frac{T}{2} \cdot 2 \cdot \cos^2\frac{x}{2} = T\cos^2\frac{x}{2}$$

$$\left(\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}\right)$$

$$\boxed{|P_{RC}|^{1/2} = \sqrt{T}\cos\frac{x}{2} = \sqrt{T}\cos\left(\frac{\pi f}{19200}\right)}$$

$$|H_c| = \left[\frac{1}{1 + \left(\frac{f}{4800}\right)^2}\right]^{1/2} \qquad \boxed{\frac{1}{|H_c|^{1/2}} = \left[1 + \left(\frac{f}{4800}\right)^2\right]^{1/4}}$$

$$\left.\begin{aligned} \therefore \; |H_t| &= \frac{A}{\alpha} G_n^{1/4} \sqrt{T}\left[1 + \left(\frac{f}{4800}\right)^2\right]^{1/4} \cdot \cos\left(\frac{\pi f}{19200}\right) \\ |H_r| &= \frac{\alpha}{G_n^{1/4}} \sqrt{T} \cdot \left[1 + \left(\frac{f}{4800}\right)^2\right]^{1/4} \cdot \cos\left(\frac{\pi f}{19200}\right) \end{aligned}\right\}$$

found my TX & RX filters

for $P_{b,min} = Q\left(\frac{A}{\sigma_n}\right) \le 10^{-6}$ $\qquad \frac{A}{\sigma_n} \ge 4.75$

$$\frac{A}{\sigma_n} = \sqrt{E_T}\left[\int_{-\infty}^{\infty} G_n^{1/2}\frac{|P_r|}{|H_c|}df\right]^{-1} = \frac{\sqrt{E_T}}{\sqrt{N_0/2}}\left[\int_{-\infty}^{\infty}\frac{|P_r|}{|H_c|}df\right]^{-1}$$

$$= \sqrt{\frac{2E_T}{N_0}}\left[\int_{-\infty}^{\infty}\frac{|P_r|}{|H_c|}df\right]^{-1}$$

$$\left[T\int_{-\infty}^{\infty}\cos^2\left(\frac{\pi f}{19200}\right)\left[1+\left(\frac{f}{4800}\right)^2\right]^{\frac{1}{2}}df\right]$$

$$= \sqrt{\frac{2E_T}{N_0}}\cdot\frac{1}{1.21}$$

$$= \frac{2}{9600}\int_0^{9600}\cos^2\left(\frac{\pi f}{19200}\right)\left[1+\left(\frac{f}{4800}\right)^2\right]^{\frac{1}{2}}df$$

$$= 1.21 \approx \int_0^1 [1+\cos(\pi x)]\left[1+(2x)^2\right]^{\frac{1}{2}}dx$$

$$= 0.83\sqrt{\frac{2E_T}{N_0}}$$

recall this is normally 1 in binary antipodal & white noise (with matched filter)

$\therefore\ -20\log 0.83 = 1.62$ dB degradation over ∞ BW case (with M.F.)

- required signal energy is...

$$E_T = \left(1.21\frac{A}{\sigma_n}\right)^2\frac{N_0}{2} = 1.21^2\cdot 4.75^2\cdot 10^{-12} = 3.3\times10^{-11}\,\text{J}$$

- avg. power is $P_T = \frac{E_T}{T} = 9600\times 3.3\times10^{-11} = 0.317\,\mu W \approx -35\,\text{dBm}$