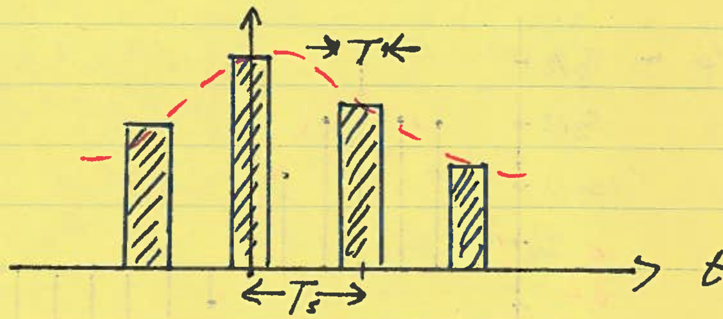
L6 Quantization & Encoding

6.1 Pulse Amplitude Modulation

- the process of (or result) of flat top sampling can more generally be viewed as a transformation of your signal...

  ... into a series of pulses with varying amplitude



- people generally refer to this as PAM and it appears in other (~~alth~~ although fundamentally identical) contexts

- in general modulation is when you multiply your signal by some other periodic waveform typically referred to as the carrier

- are we justified in calling this sampling modulation?

recall
$$x_s(t) = [\,x(t)\, x_\delta(t)\,] * p(t)$$
$$= \left[\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\right] * p(t)$$
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,p(t) * \delta(t-nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,p(t-nT_s)$$

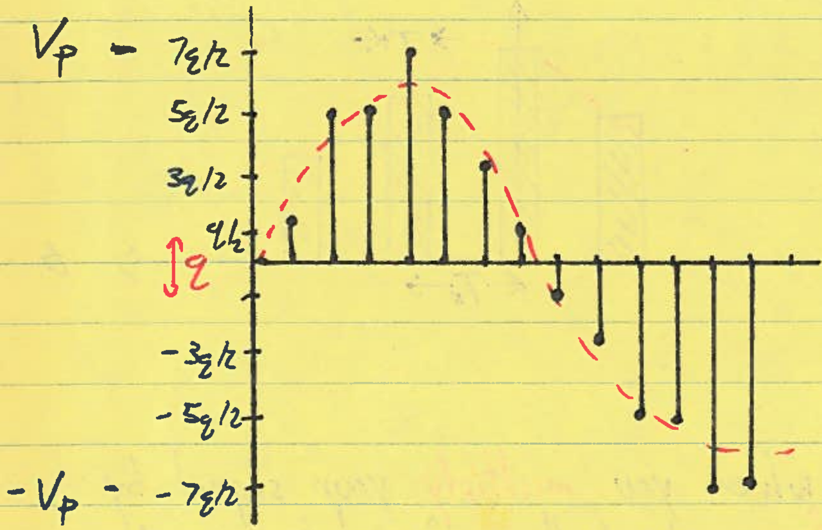
commutative
$x_1 * x_2 = x_2 * x_1$

$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t-t_0)$

- thus we are multiplying our signal by a periodic pulse train...
  ... a carrier... we are modulating → PAM

6.2 Quantizing

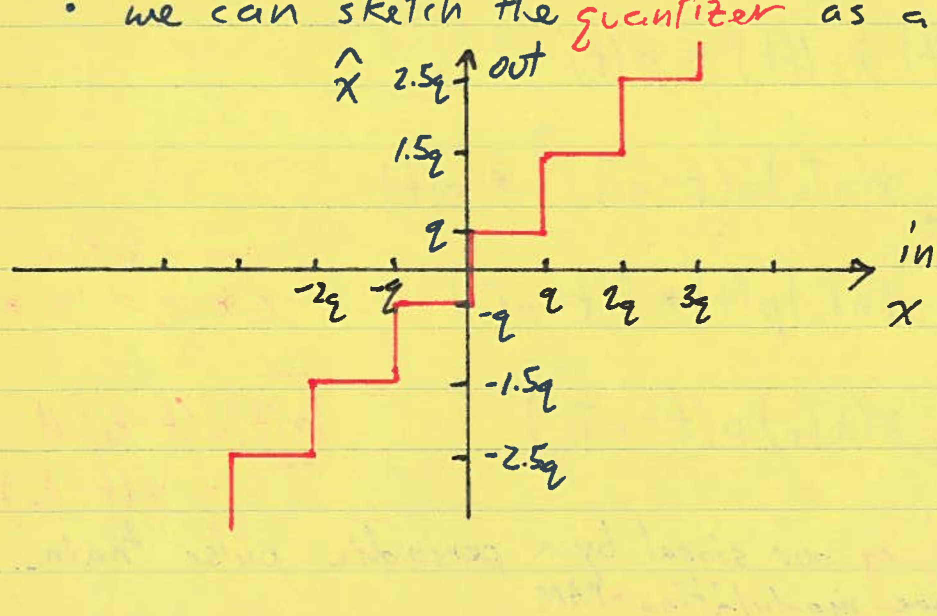
- Our sampling/PAM has resulted in discrete-time signals but with continuous amplitude range
- If you want to digitize this ... you have to approximate with one of a number of discrete levels



- say you choose a range from $-V_p$ to $+V_p$
- if you divide this into L equally spaced levels centred at zero
- the quantile interval (separation between levels) is

$$q = \frac{2V_p}{L-1} = \frac{V_{pp}}{L-1} = \frac{V_{pp}}{2^b-1} \approx \frac{V_{pp}}{2^b}$$

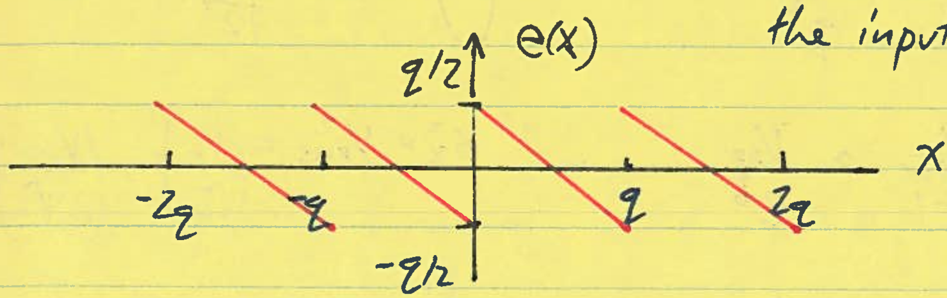
- we can sketch the quantizer as a transfer function



6.3 Quantization Error

$$\hat{x}(t) = x(t) + e(t)$$

an error dependent on the input signal and q



- mean square value (variance) of this error is a useful quantification

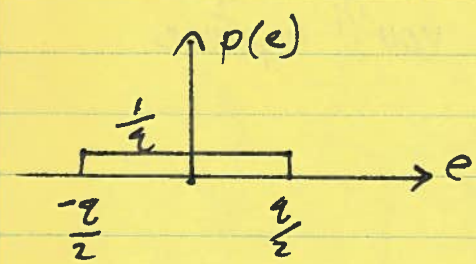
$$e(x) = -x + \frac{q}{2} \qquad e^2 = x^2 - xq + \frac{q^2}{4}$$

$$\langle e^2 \rangle = \sigma_q^2 = \frac{1}{q}\int_0^q \left(x^2 - xq + \frac{q^2}{4}\right)dx = \frac{1}{q}\left[\frac{q^3}{3} - \frac{q^3}{2} + \frac{q^3}{4}\right]$$

periodic w/ q

; assume uniform dist (i.e. $p(x) = 1/q$)

$$= \frac{q^3}{q}\left[\frac{4-6+3}{12}\right] = \frac{q^2}{12} \quad \ldots \text{ more formally} \ldots$$



$$\sigma_q^2 = \int_{-q/2}^{+q/2} e^2 p(e)\, de = \int_{-q/2}^{q/2} \frac{e^2}{q}\, de = \left.\frac{e^3}{q \cdot 3}\right|_{-q/2}^{q/2}$$

$$= \frac{1}{q}\left[\frac{q^3}{3 \cdot 8} - \frac{-q^3}{3 \cdot 8}\right] = \boxed{\frac{q^2}{12} = \sigma_q^2}$$

- In general SNR is then

$$SNR = \frac{\sigma_x^2}{\sigma_q^2} = \sigma_x^2 \cdot \frac{12}{q^2}$$

recall $q = \frac{V_{pp}}{L-1} \approx \frac{V_{pp}}{2^b}$

$$\therefore \quad SNR = \sigma_x^2 \times 12 \times \frac{2^{2b}}{V_{pp}^2}$$

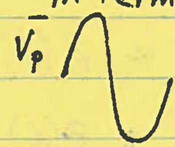
$$SNR_{dB} = \left[\log(2^{2b}) + \log 12 + \log\left(\frac{\sigma_x^2}{V_{pp}^2}\right)\right] \times 10$$

$$= 6.02\,b + 10.8 + 10\log\left(\frac{\sigma_x^2}{V_{pp}^2}\right)$$

$$= 6.02b + 1.7\cancel{3}$$ if σ_x^2 is FULL POWER

FULL POWER SIGNALS:

in terms of basic sin wave case



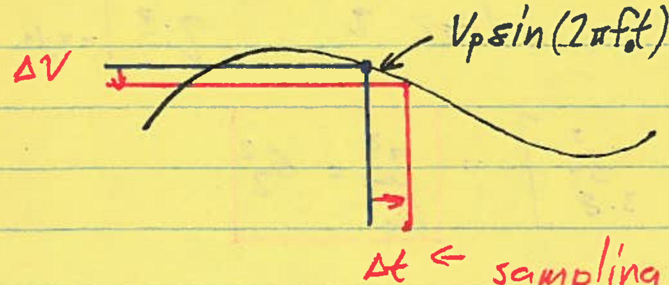
$V_{rms} = \frac{V_p}{\sqrt{2}}$

$$\sigma_x^2 = V_{rms}^2 = \left(\frac{V_p}{\sqrt{2}}\right)^2 = \left(\frac{V_{pp}}{2\sqrt{2}}\right)^2 = \frac{V_{pp}^2}{8}$$

-OR- $$SNR = \frac{V_{pp}^2}{8} \times \frac{12}{q^2} = \frac{V_{pp}^2}{8} \times \frac{12}{V_{pp}^2} \times 2^{2b} = \frac{3}{2} \times 2^{2b} = \frac{3}{2}(L-1)^2$$

6.4 Timing Jitter

- if your sampling clock moves around you'll capture the WRONG SIGNAL



$$\sigma_j^2 = E\{\Delta v^2\}$$ ... clearly signal slope is infulential so...

$$= E\left\{\left(\frac{dv}{dt}\right)^2 \cdot (\Delta t)^2\right\}$$

$$= E\left\{\left(\frac{dv}{dt}\right)^2\right\} \cdot E\{\Delta t^2\}$$

$$= E\{(V_p \times 2\pi f_0 \times \cos(2\pi f_0 t))^2\} \cdot \sigma_t^2$$

$$= \frac{V_p^2 \omega_0^2}{2} \cdot \sigma_t^2$$

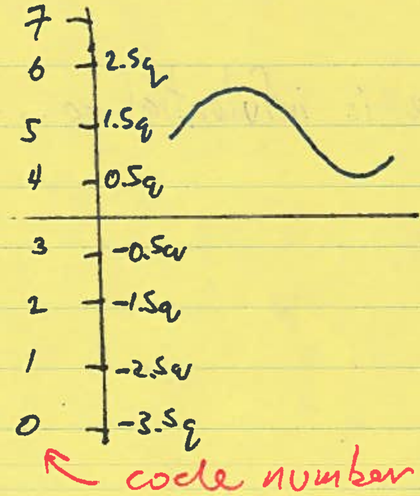
$$\therefore \text{SNR} = \frac{\sigma_x^2}{\sigma_j^2} = \frac{V_p^2}{2} \Big/ \frac{V_p^2 \omega_0^2}{2} \sigma_t^2 = \frac{1}{\omega_0^2 \sigma_t^2} = \frac{1}{4\pi^2 f_0^2 \sigma_t^2}$$

- more generally for a signal from f_L |———| f_H you can show

$$\text{SNR} = \frac{1}{\sigma_t^2} \times \frac{3}{(f_H^2 + f_H f_L + f_L^2)} \approx \frac{3}{\sigma_t^2 \cdot f_H^2}$$

6.5 Encoding

- Our quantizer produces a discrete set of voltage levels
- The job of the encoder is to convert this into a suitable digital code number



- this mapping onto some code is, in general, called PCM
- PCM code number is of course represented as a binary sequence
- basically PCM encodes each level into a digital word

- what are PCM's bandwidth requirements???
- If I use L levels ∴ need

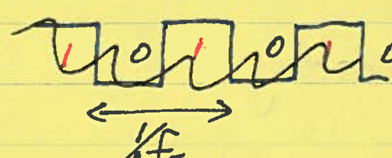
$$b = \log_2 L \text{ bits}$$

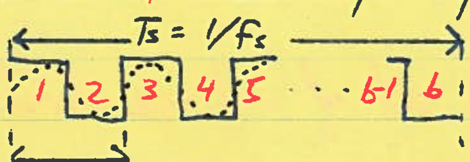
- if message BW is f_m and sampling rate is $f_s \geq 2f_m$

∴ $b f_s$ bit rate is needed ← data bandwidth

- but what's my signal bandwidth?

at worst case my digital stream is





$\frac{T_s}{b/2}$ ⇒ ∴ our highest harmonic freq. component is $\frac{b f_s}{2}$

∴ our PCM signal's bw $f_{pcm} = \frac{b f_s}{2} \geq b f_m$

→ clearly steps must be taken to compress this