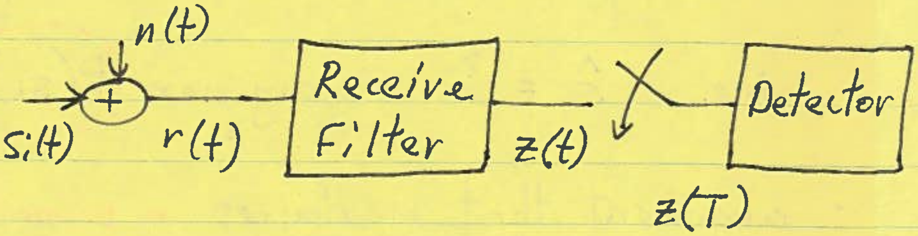
L9 The Detector

9.1 The Basic Receiver

- Recall we have (in simple form)



- assuming $h_c(t) = 1$

$$r(t) = s_i(t) + n(t) \qquad s_i(t) = \begin{cases} s_1(t) & 0 \le t \le T \text{ for } 1 \\ s_2(t) & 0 \le t \le T \text{ for } 0 \end{cases}$$

- reduce received signal to single number

$$z(T) = a_i(T) + n_o(T) \qquad i = 1, 2$$

- how do you figure out what s_i was sent? The detectors job

9.2 Basic Detector Ideas

- Let's start thinking about key ideas in a discrete symbol context
- say TX can send S messages s_i
- RX can receive R channel outputs r_j
- the probability of transmitting any one message is

$$P(s_i)$$

- before making any observation what is RX's best guess as to what message was sent

  i.e. $\hat{s} = ? = \arg\max_{s_i} P(s_i)$ ← "the prior"

- argument that maximizes a priori (before reception) probability

- and AFTER OBSERVATION ? (of r_j)

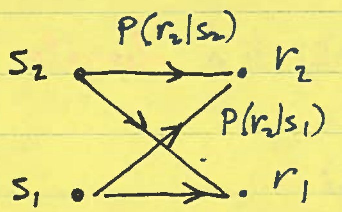
- probability that s_i was transmitted...

$$P(s_i | r_j)$$ ← a posteriori (after reception) probability

"the posterior"

- so $\hat{s}(r_j) = \arg\max_{s_i} P(s_i|r_j)$: MAP detector

- often it is much more common to model systems from input to output



BSC (Binary Symmetric Channel)

- recall Bayes rule

"conditional" OR "likelihood"

prior

posterior : prob. of message given obs.
likelihood : prob. of obs. given message

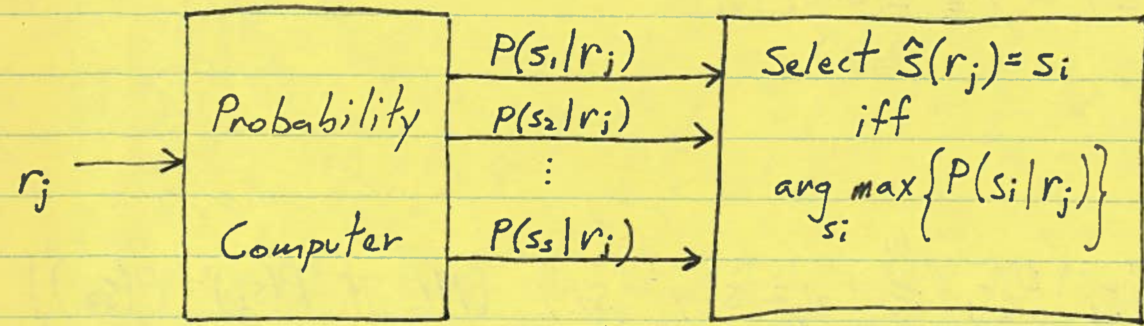
$$P(s_i|r_j) = \frac{P(r_j|s_i)\,P(s_i)}{P(r_j)}$$

"posterior"

$$\hat{s}(r_j) = \arg\max_{s_i}\left[P(r_j|s_i)\,P(s_i)\right]$$

($P(r_j)$ obviously indep. of s_i)

- thus modern receivers are probability computers



9.3 Continuous Noise Detector

- more generally say that our received signal can take on the continuous random variable

$$z(T) = a_i(T) + n_0(T)$$

- with PDF $p_z(z)$

- in the case of binary signaling our MAP detector

$$P(s_1|z) \overset{a_1}{>} P(s_2|z) \quad \text{choose } a_1$$

$$P(s_1|z) \underset{a_2}{<} P(s_2|z) \quad \text{choose } a_2$$

in case of equality either message can be chosen

- in the parlance statistical decision theory

- we call our possible "messages" or "statements" about the current state of the system (i.e. what I actually received) as "hypotheses" and write

$$P(s_1|z) \underset{H_2}{\overset{H_1}{\gtrless}} P(s_2|z) \qquad (MAP)$$

- using Bayes rule in mixed form

$$P(s_i|z) = \frac{p_z(z|s_i)\,P(s_i)}{p_z(z)}$$

- we get

$$p_z(z|s_1)P(s_1) \underset{H_2}{\overset{H_1}{\gtrless}} p_z(z|s_2)P(s_2) \quad (ML \text{ if } P(s_1) = P(s_2))$$

- assuming $z = a_i + n_0$ ← signal is distributed by additive noise (statistically)

$\cancel{p_z(z|s_i) =}$ z is received only iff $n_0 = z - a_i$

$$p_z(z|s_i) = p_n(\underbrace{z - a_i}_{\text{noise}}|s_i) = p_n(z - a_i)$$

and since n_0 is indep of s_i we have

$$p_n(z - a_1)P(s_1) \underset{H_2}{\overset{H_1}{\gtrless}} p_n(z - a_2)P(s_2)$$

- if noise is Gaussian $p_n(x) = \dfrac{e^{-x^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}}$

$$\frac{e^{-\frac{(z-a_1)^2}{2\sigma_0^2}}}{\sqrt{2\pi\sigma_0^2}}\,P(s_1) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{e^{-\frac{(z-a_2)^2}{2\sigma_0^2}}}{\sqrt{2\pi\sigma_0^2}}\,P(s_2)$$

- take natural log of both sides

$$-\frac{(z-a_1)^2}{2\sigma_0^2} + \ln(P(s_1)) \underset{H_2}{\overset{H_1}{\gtrless}} -\frac{(z-a_2)^2}{2\sigma_0^2} + \ln(P(s_2))$$

$$-(z-a_1)^2 \underset{H_2}{\overset{H_1}{\gtrless}} -(z-a_2)^2 + 2\sigma_0^2 \ln\left[\frac{P(s_2)}{P(s_1)}\right]$$

$$2z(a_1-a_2) \underset{H_2}{\overset{H_1}{\gtrless}} 2\sigma_0^2 \ln\left[\frac{P(s_2)}{P(s_1)}\right] - a_2^2 + a_1^2$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1^2-a_2^2}{2(a_1-a_2)} + \frac{\sigma_0^2}{(a_1-a_2)} \ln\left[\frac{P(s_2)}{P(s_1)}\right]$$

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1+a_2}{2} + \frac{\sigma_0^2}{a_1-a_2} \ln\left[\frac{P(s_2)}{P(s_1)}\right]$$

if $P(s_2) = P(s_1)$ (or all symbols are equally likely)

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \frac{a_1+a_2}{2} = \gamma_0 \Rightarrow$$ Maximum Likelihood Detector (a MAP detector)

9.4 MLD - Performance

- Characterize in terms of probability of error

$$P_B = P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2)$$

- for maximum likelihood detector

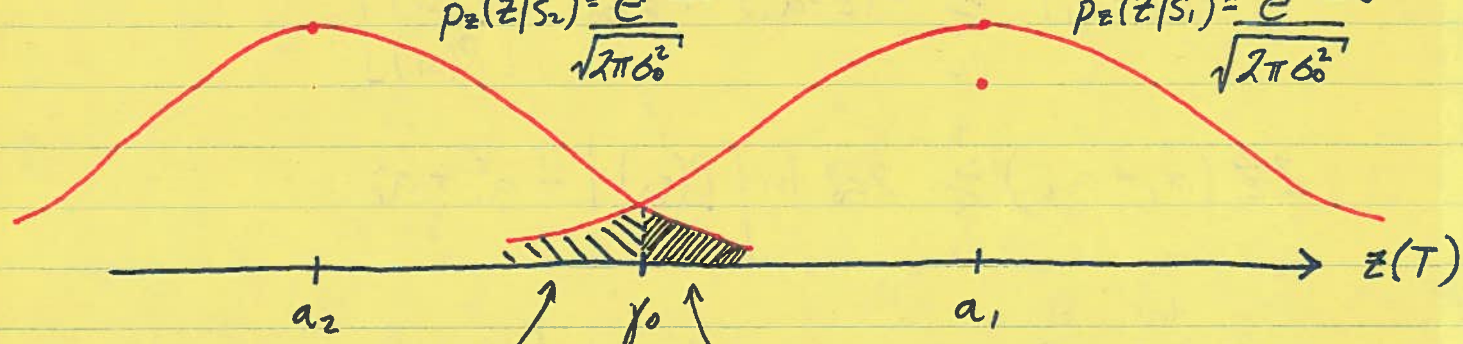
$$P_B = \frac{1}{2}\left[P(H_2|s_1) + P(H_1|s_2)\right]$$

what are these likelihoods?

- take a look at your detector's input signal & its distribution

$$z(T) = a_i(T) + n_0(T)$$

$$p_z(z|s_2) = \frac{e^{-\frac{(z-a_2)^2}{2\sigma_0^2}}}{\sqrt{2\pi\sigma_0^2}} \qquad p_z(z|s_1) = \frac{e^{-\frac{(z-a_1)^2}{2\sigma_0^2}}}{\sqrt{2\pi\sigma_0^2}}$$



$P(H_2|s_1)$ $\qquad\qquad$ $P(H_1|s_2)$

$$= \int_{-\infty}^{\gamma_0} p_z(z|s_1)\,dz \qquad\qquad = \int_{\gamma_0}^{\infty} p_z(z|S_2)\,dz$$

$$= 1 - Q\left(\frac{\gamma_0 - a_1}{\sigma_0}\right) = Q\left(\frac{a_1-\gamma_0}{\sigma_0}\right) \qquad\qquad = Q\left(\frac{\gamma_0 - a_2}{\sigma_0}\right)$$

$$P_B = \frac{1}{2}Q\left(\frac{a_1-\gamma_0}{\sigma_0}\right) + \frac{1}{2}Q\left(\frac{\gamma_0-a_2}{\sigma_0}\right) = \frac{1}{2}Q\left(\frac{a_1}{\sigma_0} - \frac{a_1+a_2}{2\sigma_0}\right) + \frac{1}{2}Q\left(\frac{a_1+a_2}{2\sigma_0} - \frac{a_2}{\sigma_0}\right)$$

$$= \frac{1}{2}Q\left(\frac{a_1-a_2}{2\sigma_0}\right) + \frac{1}{2}Q\left(\frac{a_1-a_2}{2\sigma_0}\right)$$

$$\boxed{P_B = Q\left(\frac{a_1-a_2}{2\sigma_0}\right)} \longleftarrow \text{ML-D's probability of error}$$

- How do you minimize it???

- Maximize $\left(\frac{a_1-a_2}{2\sigma_0}\right)$