York University
Department of Electrical Engineering and Computer Science
EECS 4214

Lab #2   Stochastic Signals, Filtering, and Modulation

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1 Purpose

In this lab, you will be introduced to random variables and stochastic processes. Using a computer, you will learn how to generate random variables of any power spectral density (PSD) through a linear, time invariant (LTI) system representation whose impulse response is derived from the PSD. You will also implement a quadrature amplitude modulator and demodulator as a prototype communication system.

The learners will complete the following three simulations in MATLAB and submit their solutions along with a soft copy of the code in the form of a report.

2 Objectives

By the end of this project, you will be able to:

1) Generate random variables of commonly used probability density functions (PDFs) including uniform and Gaussian PDFs.

2) Derive and plot the autocorrelation function as well as the PSD of a sequence of random variables.

3) Implement a quadrature amplitude modulator and demodulator widely used in many digital radio communications and data communications applications.

3 References

1) Handouts on "Probability and Random Variables" available on the course homepage.

2) Bernard Sklar text: Sections 1.5 - 1.7. Pages 20 – 50.

4 Introduction

Discussions of probability and statistics in engineering can be quite bewildering at first, but after some effort I think the ideas can become quite routine to the student. The following few sections try to organize the key concepts and terms used to describe statistical phenomena digital communications (and may other fields).

4.1 Random Variable Review

As covered in class, a **random variable**, $X(\omega)$, is a function (a mapping) that associates a real number with each **element**, ω, in the **sample space**, Ω, of a **random experiment** (or "experiment" for short, i.e. an action whose outcome is not certain). For example rolling a six-sided die with the intention of observing the identity of the upward facing side constitutes a random experiment. A simple random variable in this case is the mapping of each die face to a corresponding integer from 1 to 6. This mapping is pretty natural for a six-sided die, but the mapping could just as legitimately be the sequence $X(\omega) = \{-2, -1, 0, 1, 2, 3\}$.

Recall that the random variable, $X(\omega)$, is deterministic. What is random is the specific outcome (or *realization*) of that random variable, x, after any experiment.

Recall also the concept of an **event**, E_i, which is any set of possible experimental outcomes (i.e. any set of elements ω). In a discrete sample space as discussed above with N elements in the sample space there are 2^N possible events, this includes the null event $\varnothing$ (i.e. none of the possible elements happened) and the full event Ω (i.e. any of the possible elements happened). You don't have to identify all events, but you do have to identify enough events to span your sample space). Obviously when employing random variables your events will be sequences of number from the real line.

With your events identified you can now associate numbers with them which we call **probabilities**, $P(E_i)$. The associations have to follow some formal rules (e.g. $0 \leq P(E_i) \leq 1$) and physical rules (i.e. actually assigning the right numbers to your probabilities, like you reason that the chance of getting a head on a fair coin is 1/2 not 1/3).

The outcomes of an experiment do not have to be discrete of course. For example, every instant that you measure the voltage across a resistor (with an accurate enough voltmeter) you'll get a slightly different value within some range (or interval) due to the random thermal (i.e. finite temperature) agitation to which the resistor's electrons are subject. The voltage can thus be considered a **continuous random variable**. If you are dealing with a continuum of possible numbers however appreciate that the probability for seeing any one number is zero, this is a circumstance of the mathematical definitions and constructs from which the probability concept is formally defined. Only the probability of seeing a number within a certain interval can be assigned a probability. For further discussion on random variables, refer to the handout on random variables available on the course home page.

4.2 Distribution Function Review

Talking about intervals, a popular shorthand for the probability that a random variable takes on a value in the interval $(-\infty, x]$ is $F_X(x)$. When considered over the range of x this is called the **cumulative distribution function** (CDF). The probability that a random variable appears within an infinitesimally small range $\mathrm{d}x$ is determined by a term called the **probability density function** (PDF).

$$p_X(x) = f_X(x) = \frac{\mathrm{d}F_X(x)}{\mathrm{d}x}. \tag{1}$$

We often describe the statistical properties of our random variables in terms of the PDF. In class we have introduced a couple of really common ones, the **uniform** distribution $U(a,b)$ and the **Gaussian** (or **Normal**) distribution $N(\mu, \sigma^2)$.

4.3   Statistical Averages Review

In class we also discussed succinct ways to state the probabilistic characteristics of random variables in terms of their statistical averages. A simple example of these is the **expectation** or **mean** with the expectation operator $E[\cdot]$

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x. \tag{2}$$

Another example are the **nth moments** $E[X^n]$ expressed generally as

$$E[X^n] = \int_{-\infty}^{\infty} x^n p_X(x) \mathrm{d}x. \tag{3}$$

There are also the closely related **nth central moments** $E[(X - \mu_X)^n]$ of which an especially prominent one is the **variance**

$$\mathrm{Var}(X) = \sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 p_X(x) \mathrm{d}x. \tag{4}$$

4.4   Random Processes Review

There are many situations where the random experiment we choose to analyze is actually a time varying signal. You can start thinking about this in the random variable context discussed above, although now you decide to do a whole bunch of successive experiments over time and you decide to characterize your experiment in terms of these time-series. Now, you can think of your sample space as consisting of elements that vary with time. Any outcome is therefore a unique sequence (not just a unique number). In other words, instead of thinking of your outcome as coming from a random variable $X(\omega)$, we think of it as coming from a **random process** $X(\omega;t)$, a mapping to a sequence or real numbers over time, not just a mapping to a number.

The possible outcomes of $X(\omega;t)$ are called **sample functions**. For short, $X(\omega;t)$ is simply just written as $X(t_1)$ or just X_1. Examples to fortify the points are shown in Fig. 1. Therein three possible outcomes from an experiment are shown, say these represent the possible noisy voltage sequences across a resistor over time. The collection of all possible sample functions is called the **ensemble**.

Random processes are analyzed by considering the ensemble at particular moments in time. For example all the possible points of the ensemble at time t_1 can be thought of as realizations of the random variable $X(t_1) = X_1$ which has some value x_1 (the subscript 1 refers to the fact that the realization pertains to time t_1).

From this perspective we are, as usual, interested in such things as the PDF of a random variable, but now specific to a particular time. That is, we consider $p_{X_1}(x_1)$, the probability that a result x_1 is obtained at time t_1 from the random variable X_1. In general, over all time indexes, j, we would seek to extract $p_{X_j}(x_j)$. And we would extract our **ensemble averages**, such as the **mean**, in the same way as before although now we anticipate a time dependence, that is[1]

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x p_X(x;t) \mathrm{d}x \tag{5}$$

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[1] I drop the subscript on X and x here because the argument t implicitly identifies the index of the random terms.
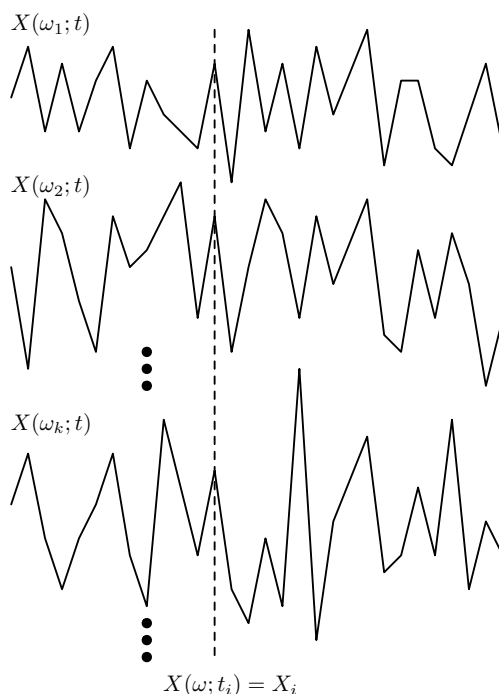
Figure 1: Random process sample functions.

Since we are talking about a multitude of random variables over time it is also of interest to express the relationship between these in the form of **joint distributions**. A simple example of such is $p_{X_1 X_2}(x_1, x_2)$[2] which represents the probability that random variable $X(\omega; t_1) = x_1$ AND ALSO that random variable $X(\omega; t_2) = x_2$. Similar relationships can be made over as many random variables from the time-series as you wish which you can generally refer to as **multivariate distributions**.

You can extract interesting ensemble averages from your multi-order distributions too, almost exactly like the mean. A really really useful ensemble average is the one that averages out the product of two random-variables from our time-series

$$R_{XX}(t_i, t_j) = E[X(t_i)X(t_j)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i x_j p_{X_i X_j}(x_i, x_j; t_i, t_j) \mathrm{d}x_i \mathrm{d}x_j \tag{6}$$

This average is also referred to as the **autocorrelation**[3]. We have already introduced this term in our discussion of deterministic signals and the expression we associated with it in that context look rather different that what I just shown above. Luckily you can show (but we won't show it) that the two expressions can be different way of saying the same thing. More on that in a moment (see the expressions (10) and (11) below), but first yet another idea to present . . .

If you find that all your multi-order distributions are the same with any time offset τ, that is you compare the statistics at different time shifts over all variables and find they are identical as in

$$p_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) = p_X(x_{1+\tau}, x_{2+\tau}, \ldots, x_{N+\tau}; t_1 + \tau, t_2 + \tau, \ldots, t_N + \tau) \tag{7}$$

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[2]You can also call it a **second-order distribution** since it functions over two random variables.

[3]The textbook expresses the autocorrelation with a symbol consisting of only one subscript X, just giving you a heads up so that you don't get too confused. A lot of people use the double subscript XX that I employ in this document.

you refer to your random process as **strict sense stationary** (SSS).

A weaker, but extremely useful condition is one where

$$m_X(t) = m_X \tag{8}$$
$$R_{XX}(t_i, t_j) = R_{XX}(t_i, t_i + \tau) = R_{XX}(\tau) \tag{9}$$

This condition says that your process has a constant mean (i.e. does not change with time) and that its autocorrelation is dependent only on the time difference between random variables (i.e. not dependent on absolute time). You refer to such a random process as **wide sense stationary** (WSS). From an analysis point of view, this is a convenient property for a random signal to possess (i.e. we don't have to do as much work to analyze it).

Another helpful property that we will assume throughout the course is **egrodicity**. This basically means that if we observe a signal long enough it will demonstrate all the sequences that we would observe over the whole ensemble of the sample functions. So instead of having to find p_X as used in (5) and (6) and use double integration and all that complicated stuff we can employ the (slightly less complicated I hope!) **time averages**

$$m_X = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \mathrm{d}t \tag{10}$$

$$R_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t + \tau) \mathrm{d}t \tag{11}$$

Note that in the above I have assumed the signal is not only ergodic but also WSS, the two don't have to occur together but in this course we will assume all our random processes have both these properties.

4.5 Correlated Random Processes (Not a Review)

One final thing. We can expand our discussion of statistical averages a bit to consider relations between different random variables X and Y. For example the (k, n)th moment of these random variables can be expressed as

$$m_{kn} = E[X^k Y^n] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^n p_{XY}(x, y) \mathrm{d}x \mathrm{d}y \tag{12}$$

The (1,1)th moment is called the **correlation** (obviously a very similar idea to the autocorrelation we have considered above)

$$m_{11} = E[XY]. \tag{13}$$

The **covariance** is the same idea, except with the means removed

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X]E[Y]. \tag{14}$$

If $\mathrm{Cov}(X, Y) = 0$ we say that X and Y are **uncorrelated**. If two (or more) random variables are uncorrelated and have the same probability distributions we say they are **independent identically distributed** (i.i.d.).

The degree to which two random variables are related is called the **correlation coefficient** and is usually expressed at

$$\rho(X, Y) = \rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \tag{15}$$

It can be shown that $-1 \le \rho_{XY} \le 1$. When $\rho_{XY} = 1$ the random variables are identical. When $\rho_{XY} = -1$ the random variables are identical except with a sign flip around their means. In general you can establish the random variable relation

$$Y = \rho_{XY} X + \sqrt{1 - \rho_{XY}^2} Z \tag{16}$$

where Z is another random variable and where Z and X are i.i.d.

More generally, people often use a $n \times n$ covariance matrix expression to express the relationship between different n random variables

$$C = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \rho_{n3}\sigma_n\sigma_3 & \cdots & \sigma_n^2 \end{bmatrix} \tag{17}$$

where I use the subscripts $1, 2, \ldots, n$ to stand for different random variables $X_1, X_2, \ldots X_n$.

Besides an expression like (16) you can indirectly use C to generate correlated random values by first noting the expression

$$C = R^T R \tag{18}$$

where R is an upper triangular matrix obtained by the process of Cholesky factorization (see `chol` in your friendly MATLAB package) and T denotes the matrix transpose. You can then express a set of correlated random variables y (a $n \times 1$ matrix of random variables) using

$$y = Rx \tag{19}$$

where R is an upper triangular $n \times n$ matrix derived from your covariance matrix as just described above and x is a $n \times 1$ matrix of uncorrelated random variables.

So what's all this got to do with random processes? Remember random processes are just collections of random variables accrued over time. So to come up with correlated random processes just apply the mechanisms described in this sub-section to each of your random process samples.

5 Generation of a Gaussian Sequence

In the first simulation, you are required to generate 1000 pairs of Gaussian random variables X_1 and X_2 that have mean vector

$$m \quad = \quad E[x] = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{20}$$

where

$$x = [X_1, X_2]^T \tag{21}$$

Assume that the covariance matrix for these random variables is

$$C \quad = \quad \begin{bmatrix} 2.5 & 1.7 \\ 1.7 & 1.5 \end{bmatrix}. \tag{22}$$

Problem 1 In MATLAB, the function `randn` generates random variables with Gaussian distributions of $N(0,1)$ (i.e. mean zero and variance one). Write a function based on `randn` that will generate the correlated random variables X_1 and X_2 defined above. Show your code (for full marks make sure the code is very clearly annotated with comments) and state what the correlation coefficient between the two random variables (RVs) is.

Problem 2 For the random variables generated above, approximate the mean of 500 samples, $i = 1, 2, \ldots, 500$ from each random variable, using the relationships

$$\hat{m}_1 \;=\; \frac{1}{500} \sum_{i=1}^{500} x_{1i} \tag{23}$$

$$\hat{m}_2 \;=\; \frac{1}{500} \sum_{i=1}^{500} x_{2i}. \tag{24}$$

Note that x_{1i} refers to an ith sample of the X_1 random variable (and same idea for x_2). The ^ above the mean symbols implies that you are only estimating the true means. For this problem you just need show your calculated estimates.

Problem 3 Compare the values obtained from the samples you processed in the problem above with the theoretical values we intended to achieve in (20). Why are the two different? Now, increase the number of samples to 10000. Does this affect your answer in any way? Has the approximation improved?

Problem 4 Now calculate the covariances (the variance is just a special case of the covariance), σ_1^2, σ_2^2, σ_{12} using 1000 points (not 500). The analytical expression for the covariances are

$$\text{Variance of } X_1: \quad \sigma_1^2 = \frac{1}{1000} \sum_{i=1}^{1000} (x_{1i} - m_1)^2 \tag{25}$$

$$\text{Variance of } X_2: \quad \sigma_2^2 = \frac{1}{1000} \sum_{i=1}^{1000} (x_{2i} - m_2)^2 \tag{26}$$

$$\text{Covariance of } X_1 \text{ and } X_2: \quad \sigma_{12} = \frac{1}{1000} \sum_{i=1}^{1000} (x_{1i} - m_1)(x_{2i} - m_2). \tag{27}$$

6   Filtering of Stochastic Signals

Problem 5 Modify the code of **Problem 1** to generate an independent identically distributed (i.i.d.) sequence $\{x_n\}$ of $L = 1000$ uniformly (i.e. not Gaussian) distributed random variables in the interval [-0.5, 0.5]. Unlike **Problem 1**, you are now dealing with a single random variable with uniform distribution in the interval [-0.5, 0.5]. Calculate the mean and variance of the generated sequence and compare with the theoretical values.

Problem 6 Compute and plot the autocorrelation function $R_X[m]$ and the power spectral density $G_X(f)$ using the following relations:

$$R_X[m] \;=\; \frac{1}{N-m} \sum_{n=1}^{N-m} x[n]x[n+m], \quad \text{for } m = 0, 1, \ldots, 50 \tag{28}$$

$$R_X[m] \;\xleftrightarrow{\;\mathscr{F}\;}\; G_X[k]. \tag{29}$$

Make sure to the plot autocorrelation function $R_X[m]$ and the power spectrum $G_X[k]$ for different ensembles of $x[n]$. You should see that the two exhibit a significant variability over different ensembles. To get reasonable results, it is necessary to average the sample autocorrelation and power spectrum over several ensembles.

Problem 7 The sequence $\{x_n\}$ that you just generated is to be passed through a causal and stable linear filter with impulse response

$$h[n] \quad = \quad \begin{cases} (0.95)^n, & \text{for } n \geq 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{30}$$

Prove that the recursive equation that describes the output $y[n]$ of this filter as a function of the input $x[n]$ is

$$y[n] \quad = \quad 0.95y[n-1] + x[n] \quad \text{for } n \geq 0 \text{ and } y[-1] = 0. \tag{31}$$

Problem 8 The filter described above produces an output with more energy than the input. If we were interested in a non-active (i.e. passive) filter which preserves the input energy at the output, what should we scale the output by? Show your work.

Problem 9 Using (31) and the `filter` function in MATLAB, generate the random sequence $y[n]$ that results if the input sequence $\{x_n\}$ is filtered by the linear filter. Compute and plot the autocorrelation function $R_Y[m]$ and the power spectral density $G_Y(f)$ for the output sequence $\{y_n\}$. Is there any relationship between the input and output power spectral densities? Why or why not? If yes, prove the relationship is indeed valid for the above simulation.

7   Quadrature Amplitude Modulation

The following problems serve as an early look at the idea of bandpass modulation, the multiplication of your information signal by a periodic (usually harmonic) waveform.

Problem 10 Using the code of **Problem 5**, generate two i.i.d. sequences $\{w_{cn}\}$ and $\{w_{sn}\}$ of $N = 1000$ uniformly distributed random numbers in the interval [-0.5, 0.5]. Compare their respective means and variances with each other and with the theoretical values.

Problem 11 Each of these sequences is passed through a linear filter with impulse response

$$h[n] \quad = \quad \begin{cases} (0.5)^n, & \text{for } n \geq 0 \\ 0 & \text{elsewhere.} \end{cases} \tag{32}$$

Prove that the input-output characteristic of the above LTI system, is given by the recursive relation

$$x[n] \quad = \quad 0.5x[n-1] + w[n] \quad \text{for } n \geq 1 \text{ and } x[0] = 0. \tag{33}$$

where $x[n]$ is now the output and $w[n]$ the input.

Problem 12 Calculate the resulting output sequences, $\{x_{cn}\}$ and $\{x_{sn}\}$ when $\{w_{cn}\}$ and $\{w_{sn}\}$ are applied at the input of the system in **Problem 11**. After the filter take your output sequences and make $\{x_{cn}\}$ modulate the (so called "in-phase") carrier $\cos(\pi/2)n$ and make the output sequence $\{x_{sn}\}$ modulate the (so called "quadrature") carrier $\sin(\pi/2)n$. You have multiplied your sequences by sinusoids and thus have shifted their spectra in the frequency domain. The net bandpass signal is formed by combining the modulated components as in

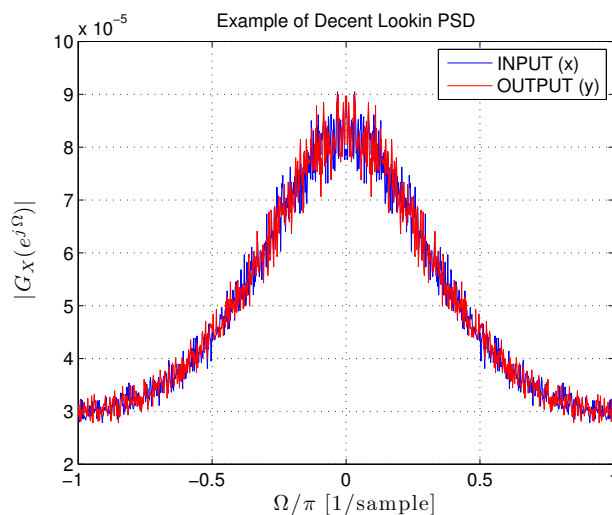$$z[n] \quad = \quad x_{cn}\cos\left(\frac{\pi}{2}n\right) + x_{sn}\sin\left(\frac{\pi}{2}n\right). \tag{34}$$

Figure 2: PSD example.

Plot the autocorrelation functions $R_{Xc}[m]$ and $R_{Xs}[m]$ for $|m| \leq 10$ for the sequences $\{x_{cn}\}$ and $\{x_{sn}\}$ respectively. Also, compute the autocorrelation function $R_Z[m]$ for $|m| \leq 10$ for the net bandpass signal $z[n]$. Is there any relationship between the three autocorrelation functions? As in **Problem 9**, you are required to average the sample autocorrelations over several realizations (i.e. ensembles).

Problem 13 Use the DFT (calculated using `fft` function in MATLAB) to compute the power spectra $G_{Xc}(e^{j\Omega})$, $G_{Xs}(e^{j\Omega})$, and $G_Z(e^{j\Omega})$. Plot the power spectra and comment on the results. Re-adjust your previous signal settings to form spectra with enough resolution and small variance (i.e. the spectra should not look too "noisy"). As before at least one thing you'll need to do is to average over ensembles. An example of what I think would be a sufficiently good looking spectrum (obviously not the signal you have to show) as far as resolution and smoothness go is shown in Fig. 2.