Linear Temporal Logic

EECS 4315

www.cse.yorku.ca/course/4315/

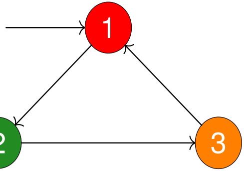
Definition

The relation $\models$ is defined by

$$
\begin{array}{rcl}
\pi \models \text{true} & & \\
\pi \models a & \text{iff} & a \in \ell(\pi[0]) \\
\pi \models \varphi \wedge \psi & \text{iff} & \pi \models \varphi \text{ and } \pi \models \psi \\
\pi \models \neg\varphi & \text{iff} & \text{not}(\pi \models \varphi) \\
\pi \models \bigcirc\varphi & \text{iff} & \pi[1..] \models \varphi \\
\pi \models \varphi \,\mathsf{U}\, \psi & \text{iff} & \exists i \geq 0 : \pi[i..] \models \psi \text{ and } \forall 0 \leq j < i : \pi[j..] \models \varphi
\end{array}
$$

$\pi \models \varphi$: the execution path π satisfies the LTL formula φ.

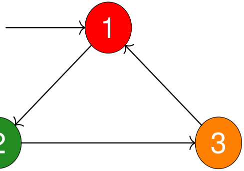
Semantics of LTL



Question

$123123\cdots \models$ green?

Semantics of LTL



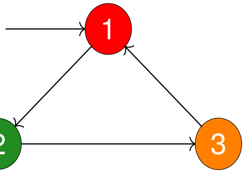
Question

$123123\cdots \models \text{green}$?

Answer

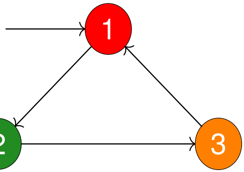
$$
\begin{array}{ll}
 & 123123\cdots \models \text{green} \\
\text{iff} & \text{green} \in \ell(1) \\
\text{iff} & \text{false}
\end{array}
$$

Semantics of LTL



Question

$123123\cdots \models \bigcirc \text{green}$?



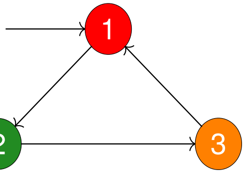
Question

$123123\cdots \models \bigcirc \text{green}$?

Answer

$$\begin{array}{ll} & 123123\cdots \models \bigcirc \text{green} \\ \text{iff} & 23123\cdots \models \text{green} \\ \text{iff} & \text{green} \in \ell(2) \\ \text{iff} & \text{true} \end{array}$$

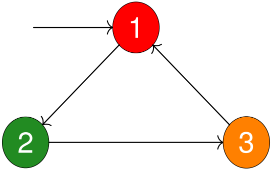
Semantics of LTL



Question

$123123 \cdots \models \text{red} \wedge \bigcirc \text{green}$?

Semantics of LTL



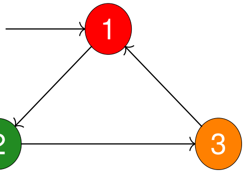
Question

$123123 \cdots \models \text{red} \wedge \bigcirc \text{green}$?

Answer

$$\begin{array}{ll} & 123123 \cdots \models \text{red} \wedge \bigcirc \text{green} \\ \text{iff} & (123123 \cdots \models \text{red}) \text{ and } (123123 \cdots \models \bigcirc \text{green}) \\ \text{iff} & (123123 \cdots \models \text{red}) \text{ and } (23123 \cdots \models \text{green}) \\ \text{iff} & (\text{red} \in \ell(1)) \text{ and } (\text{green} \in \ell(2)) \\ \text{iff} & \text{true} \end{array}$$

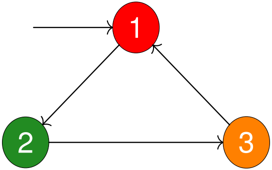
Semantics of LTL



Question

$123123\cdots \models \neg$green?

Semantics of LTL



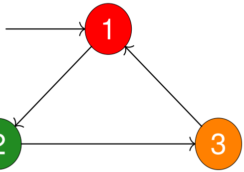
Question

$123123\cdots \models \neg \text{green}$?

Answer

$$\begin{array}{ll} & 123123\cdots \models \neg\text{green} \\ \text{iff} & \text{not}(123123\cdots \models \text{green}) \\ \text{iff} & \text{not}(\text{green} \in \ell(1)) \\ \text{iff} & \text{true} \end{array}$$

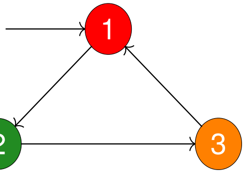
Semantics of LTL



Question

$123123\cdots \models \text{red}\ \mathsf{U}\ \text{green}$?

Semantics of LTL



Question

$123123\cdots \models$ red U green?

Answer

$$
\begin{array}{ll}
 & 123123\cdots \models \text{red U green} \\
\text{iff} & \exists i \geq 0 : 123123\cdots[i..] \models \text{red and} \\
 & \forall 0 \leq j < i : 123123\cdots[j..] \models \text{green}
\end{array}
$$

Semantics of LTL

$$\begin{array}{ll} & 123123\cdots \models \text{red}\ \mathsf{U}\ \text{green} \\ \text{iff} & \exists i \geq 0 : 123123\cdots[i..] \models \text{green and} \\ & \forall 0 \leq j < i : 123123\cdots[j..] \models \text{red} \end{array}$$

Choose $i = 1$.

$$\begin{array}{ll} & 123123\cdots[1..] \models \text{green and} \\ & \forall 0 \leq j < 1 : 123123\cdots[j..] \models \text{red} \\ \text{iff} & 23123\cdots \models \text{green and } 123123\cdots[0..] \models \text{red} \\ \text{iff} & 23123\cdots \models \text{green and } 123123\cdots \models \text{red} \\ \text{iff} & \text{green} \in \ell(2) \text{ and red} \in \ell(1) \\ \text{iff} & \text{true} \end{array}$$

Eventually

Question

$\pi \models \Diamond\varphi$ iff?

Eventually

Question

$\pi \models \Diamond\varphi$ iff?

Answer

$$
\begin{array}{ll}
 & \pi \models \Diamond\varphi \\
\text{iff} & \pi \models \text{true U } \varphi \\
\text{iff} & \exists i \geq 0 : \pi[i..] \models \varphi \text{ and } \forall 0 \leq j < i : \pi[j..] \models \text{true} \\
\text{iff} & \exists i \geq 0 : \pi[i..] \models \varphi
\end{array}
$$

Always

Question

$\pi \models \Box\varphi$ iff?

Question

$\pi \models \Box\varphi$ iff?

Answer

$$
\begin{array}{ll}
 & \pi \models \Box\varphi \\
\text{iff} & \pi \models \neg\Diamond\neg\varphi \\
\text{iff} & \text{not}(\pi \models \Diamond\neg\varphi) \\
\text{iff} & \text{not}(\exists i \geq 0 : \pi[i..] \models \neg\varphi) \\
\text{iff} & \forall i \geq 0 : \text{not}(\pi[i..] \models \neg\varphi) \\
\text{iff} & \forall i \geq 0 : \pi[i..] \models \varphi
\end{array}
$$

Infinitely Often

Question

$\pi \models \Box\Diamond\varphi$ iff?

Infinitely Often

Question

$\pi \models \Box\Diamond\varphi$ iff?

Answer

$$
\begin{array}{ll}
& \pi \models \Box\Diamond\varphi \\
\text{iff} & \forall i \geq 0 : \pi[i..] \models \Diamond\varphi \\
\text{iff} & \forall i \geq 0 : \exists j \geq 0 : \pi[i..][j..] \models \varphi \\
\text{iff} & \forall i \geq 0 : \exists j \geq 0 : \pi[(i+j)..] \models \varphi \\
\text{iff} & \forall i \geq 0 : \exists h \geq i : \pi[h..] \models \varphi
\end{array}
$$

Almost All

Question

$\pi \models \Diamond \Box \varphi$ iff?

Almost All

Question

$\pi \models \Diamond\Box\varphi$ iff?

Answer

$$\begin{array}{ll} & \pi \models \Diamond\Box\varphi \\ \text{iff} & \exists i \geq 0 : \pi[i..] \models \Box\varphi \\ \text{iff} & \exists i \geq 0 : \forall j \geq 0 : \pi[i..][j..] \models \varphi \\ \text{iff} & \exists i \geq 0 : \forall j \geq 0 : \pi[(i+j)..] \models \varphi \\ \text{iff} & \exists i \geq 0 : \forall h \geq i : \pi[h..] \models \varphi \end{array}$$

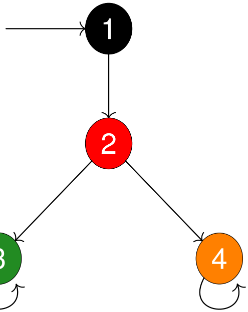
Semantics of LTL

$$TS \models \varphi \text{ iff } \forall s \in I : s \models \varphi$$

where

$$s \models \varphi \text{ iff } \forall \pi \in Paths(s) : \pi \models \varphi$$

Semantics of LTL



Prove

- $TS \models \text{true}$
- $TS \models \bigcirc\bigcirc(\text{green} \vee \text{orange})$
- $TS \models \Diamond(\text{green} \vee \text{orange})$

$TS \models \bigcirc\bigcirc(\text{green} \vee \text{orange})$

$$\begin{array}{ll} & 1233\cdots \models \bigcirc\bigcirc(\text{green} \vee \text{orange}) \\ \text{iff} & 233\cdots \models \bigcirc(\text{green} \vee \text{orange}) \\ \text{iff} & 33\cdots \models \text{green} \vee \text{orange} \\ \text{iff} & (33\cdots \models \text{green}) \text{ or } (33\cdots \models \text{orange}) \\ \text{iff} & (\text{green} \in \ell(3)) \text{ or } (\text{orange} \in \ell(3)) \\ \text{iff} & \text{true} \end{array}$$

Similarly, we can show that $1244\cdots \models \bigcirc\bigcirc(\text{green} \vee \text{orange})$.

Equivalence

Definition

LTL formulas φ and ψ are *equivalent*, denoted $\varphi \equiv \psi$, if $TS \models \varphi$ iff $TS \models \psi$ for all transition systems TS.

Equivalence

Prove

- $\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$
- $\neg \Diamond \varphi \equiv \Box \neg \varphi$
- $\neg \Box \varphi \equiv \Diamond \neg \varphi$
- $\Diamond \Diamond \varphi \equiv \Diamond \varphi$
- $\Box \Box \varphi \equiv \Box \varphi$
- $\varphi \mathsf{U} (\varphi \mathsf{U} \psi) \equiv \varphi \mathsf{U} \psi$
- $(\varphi \mathsf{U} \psi) \mathsf{U} \psi \equiv \varphi \mathsf{U} \psi$
- $\varphi \mathsf{U} \psi \equiv \psi \vee (\varphi \wedge \bigcirc (\varphi \mathsf{U} \psi))$
- $\Diamond \varphi \equiv \varphi \vee \bigcirc \Diamond \varphi$
- $\Box \varphi \equiv \varphi \wedge \bigcirc \Box \varphi$

Equivalence

$$\neg \bigcirc \varphi \equiv \bigcirc \neg \varphi$$

$$\begin{array}{ll} & \pi \models \neg \bigcirc \varphi \\ \text{iff} & \text{not}(\pi \models \bigcirc \varphi) \\ \text{iff} & \text{not}(\pi[1..] \models \varphi) \\ \text{iff} & \pi[1..] \models \neg \varphi \\ \text{iff} & \pi \models \bigcirc \neg \varphi \end{array}$$

Quiz on Thursday

- Quiz will cover the reading material of March 1 and 7.
- Quiz will be 30 minutes.
- Definitions will be provided.