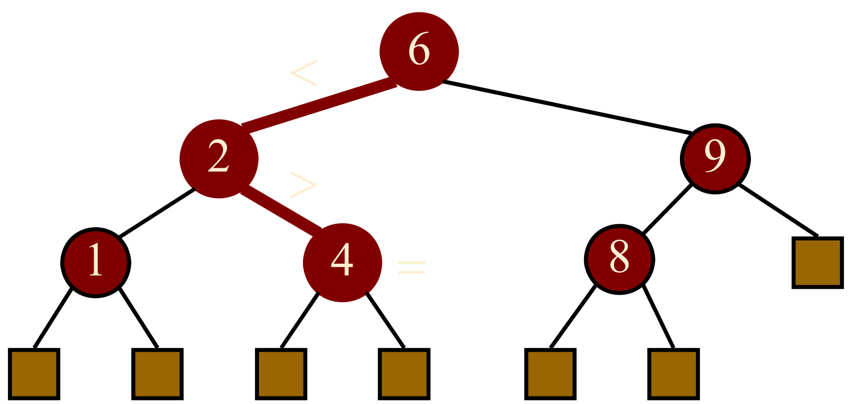
Search Trees

Chapter 11



Outline

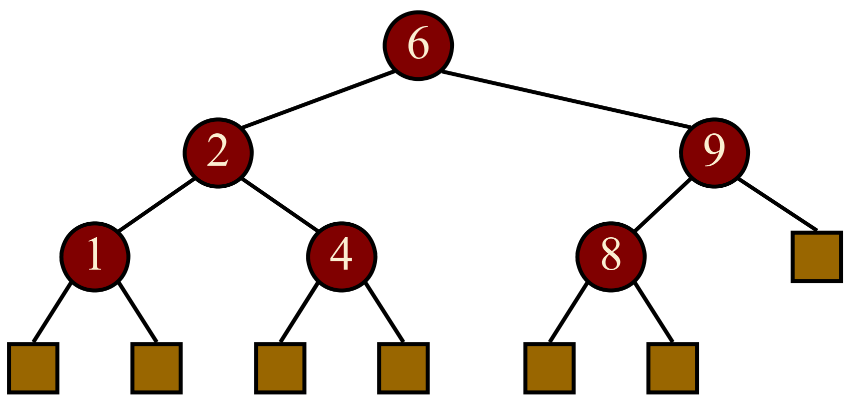
- Binary Search Trees
- AVL Trees
- Splay Trees

Outline

- **Binary Search Trees**
- AVL Trees
- Splay Trees

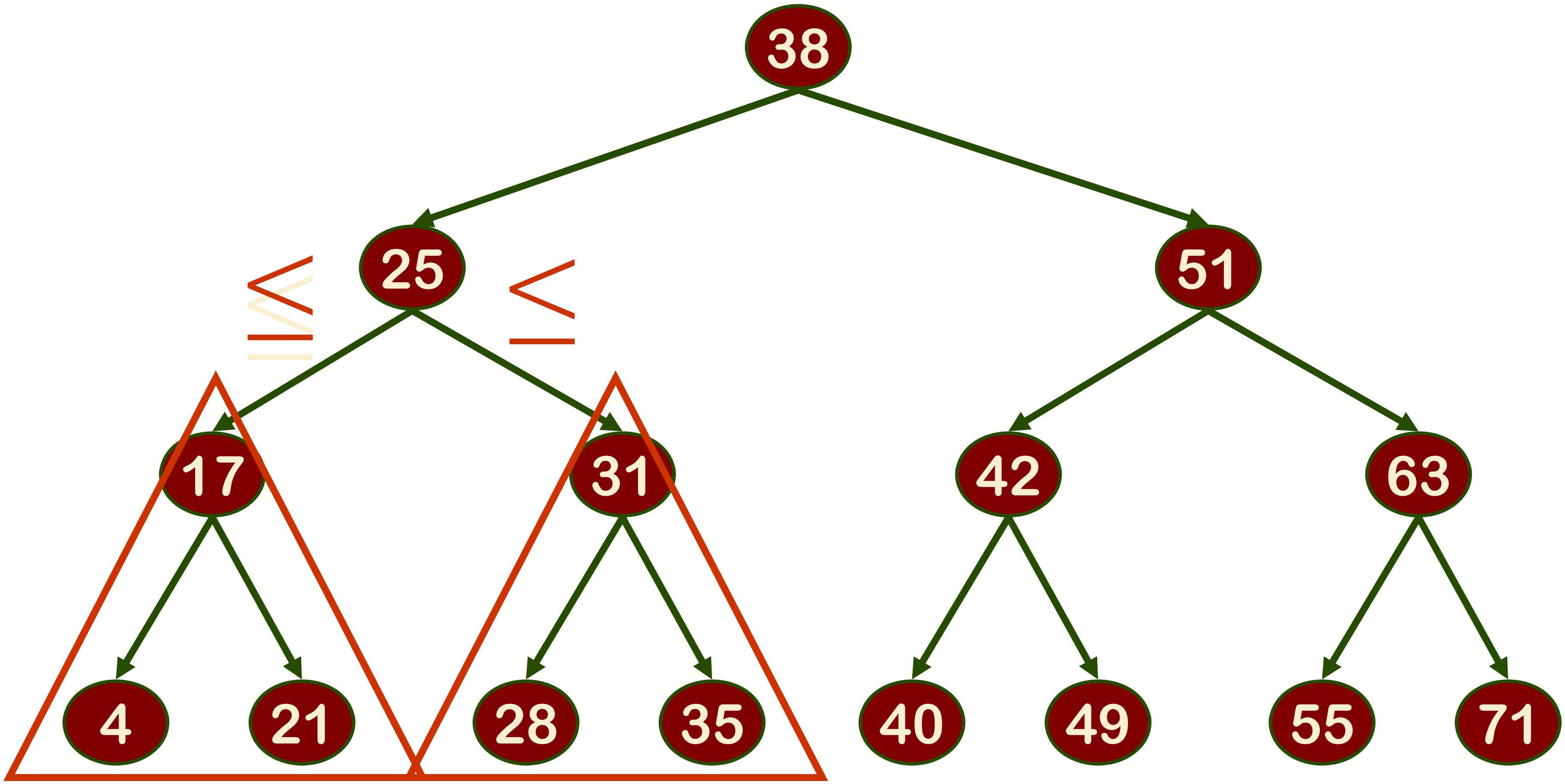
Binary Search Trees

- A binary search tree is a **proper** binary tree storing key-value entries at its internal nodes and satisfying the following property:
  - Let u, v, and w be three nodes such that u is in the left subtree of v and w is in the right subtree of v. We have $key(u) \le key(v) \le key(w)$
- We will assume that external nodes are 'placeholders': they do not store entries (makes algorithms a little simpler)
- An inorder traversal of a binary search trees visits the keys in increasing order
- Binary search trees are ideal for maps or dictionaries with ordered keys.



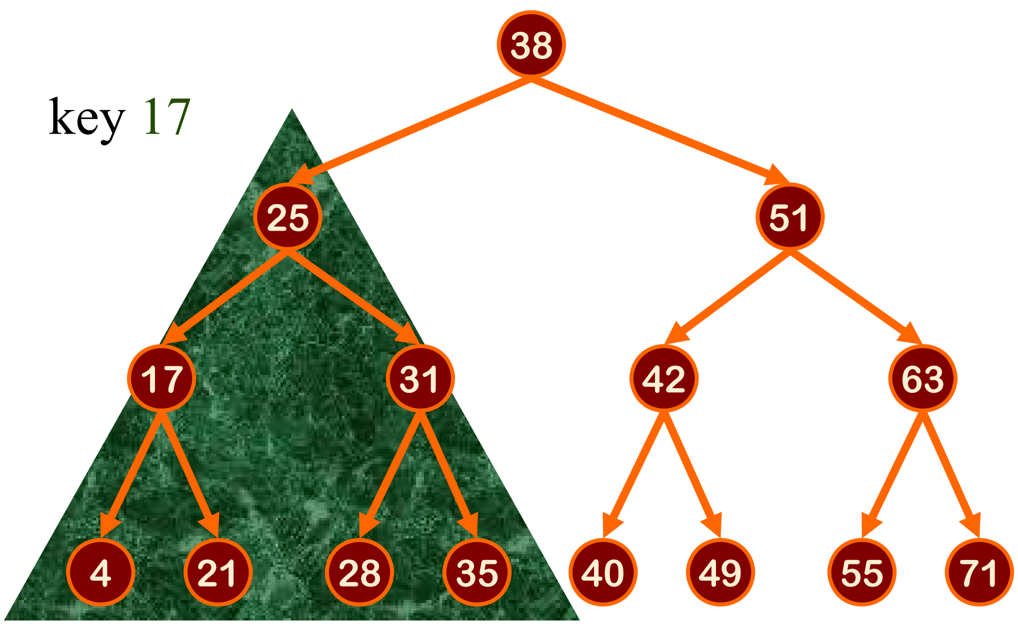
Binary Search Tree

All nodes in left subtree $\leq$ Any node $\leq$ All nodes in right subtree



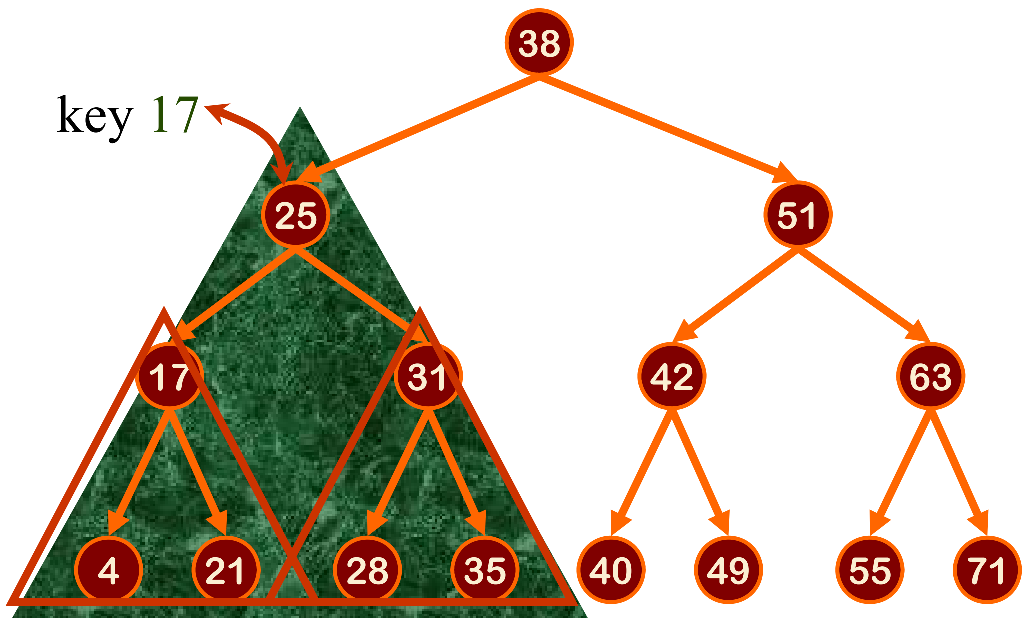
Search: Loop Invariant

- Maintain a sub-tree.
- If the key is contained in the original tree, then the key is contained in the sub-tree.



Search: Define Step

- Cut sub-tree in half.
- Determine which half the key would be in.
- Keep that half.



If key < root, then key is in left half.

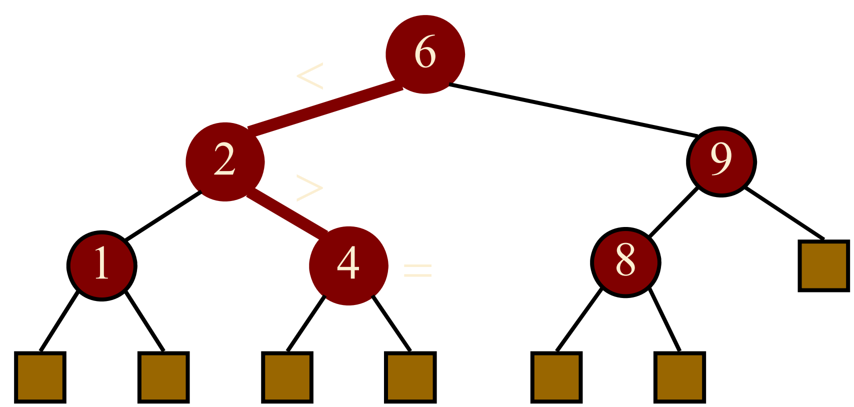
If key = root, then key is found

If key > root, then key is in right half.

Search: Algorithm

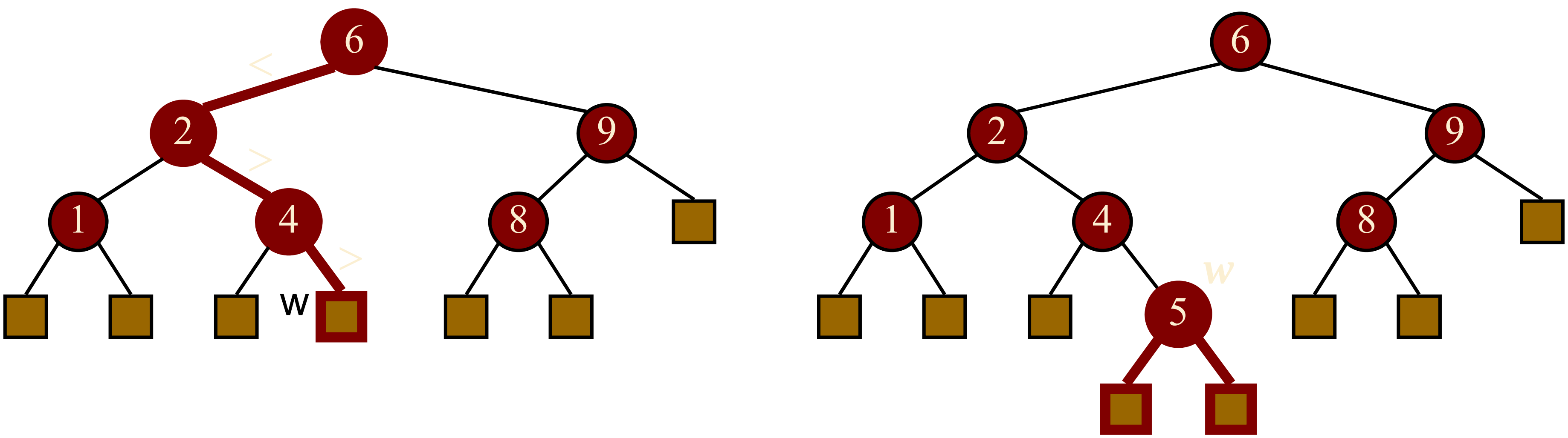
- To search for a key ***k***, we trace a downward path starting at the root
- The next node visited depends on the outcome of the comparison of ***k*** with the key of the current node
- If we reach a leaf, the key is not found and return of an external node signals this.
- Example: find(4):
  - Call TreeSearch(4,root)

Algorithm ***TreeSearch***(***k***, ***v***)
    if ***T.isExternal*** (***v***)
        return ***v***
    if ***k*** < ***key***(***v***)
        return ***TreeSearch***(***k***, ***T.left***(***v***))
    else if ***k*** = ***key***(***v***)
        return ***v***
    else { ***k*** > ***key***(***v***) }
        return ***TreeSearch***(***k***, ***T.right***(***v***))



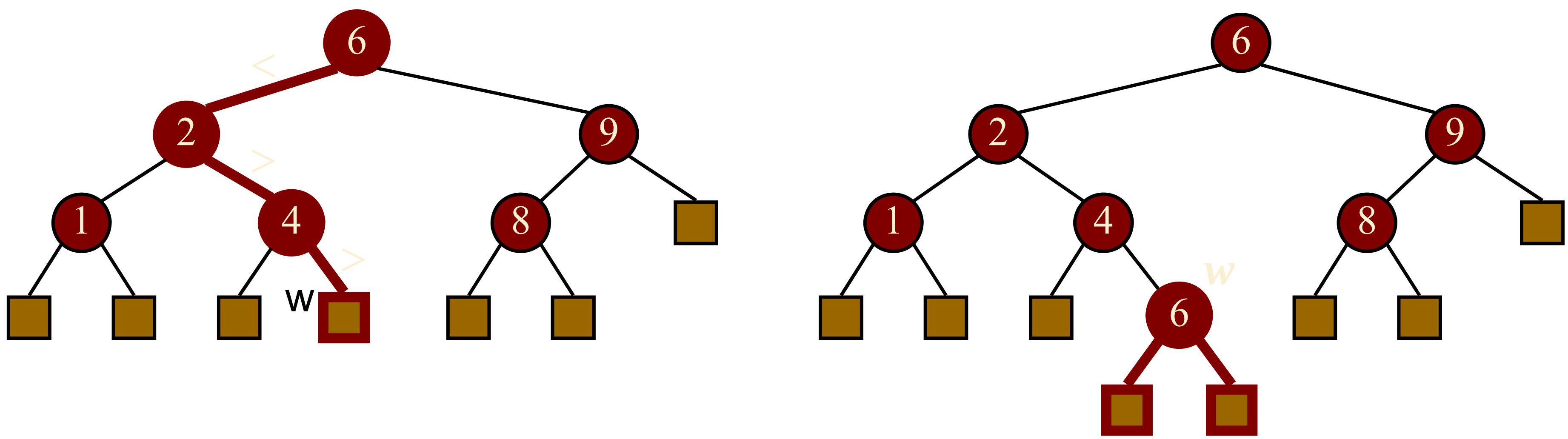
Insertion

- To perform operation insert(k, o), we search for key **k** (using TreeSearch)
- Suppose **k** is not already in the tree, and let **w** be the leaf reached by the search
- We insert **k** at node **w** and expand **w** into an internal node
- Example: insert 5



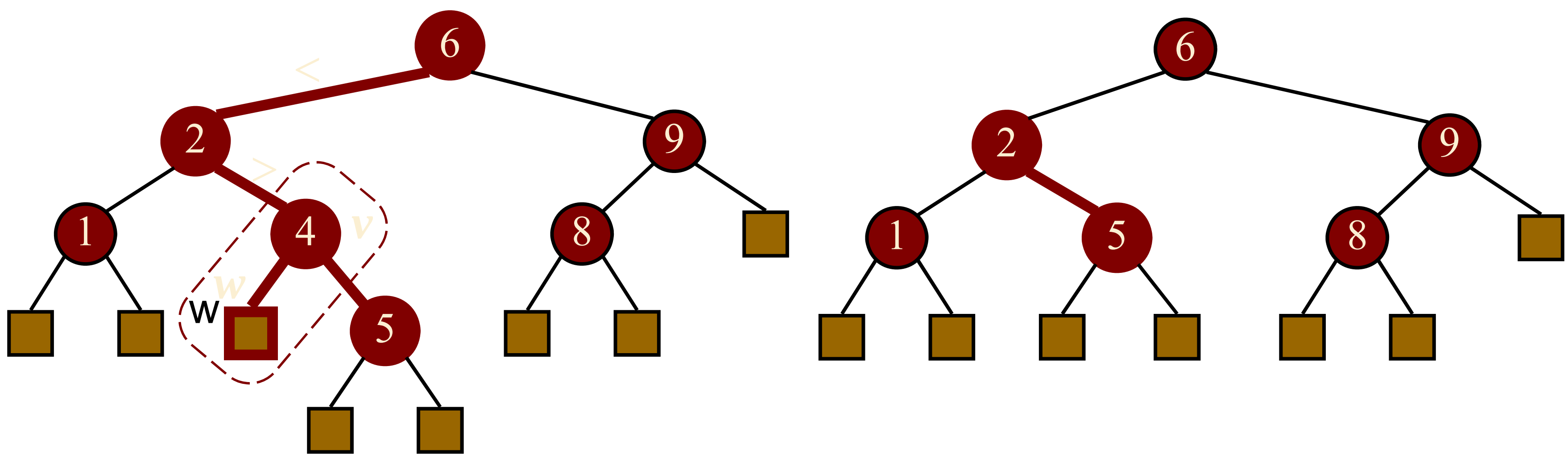
Insertion

- Suppose **k is** already in the tree, at node **v**.
- We continue the downward search through **v**, and let **w** be the leaf reached by the search
- Note that it would be correct to go either left or right at **v**. We go left by convention.
- We insert **k** at node **w** and expand **w** into an internal node
- Example: insert 6



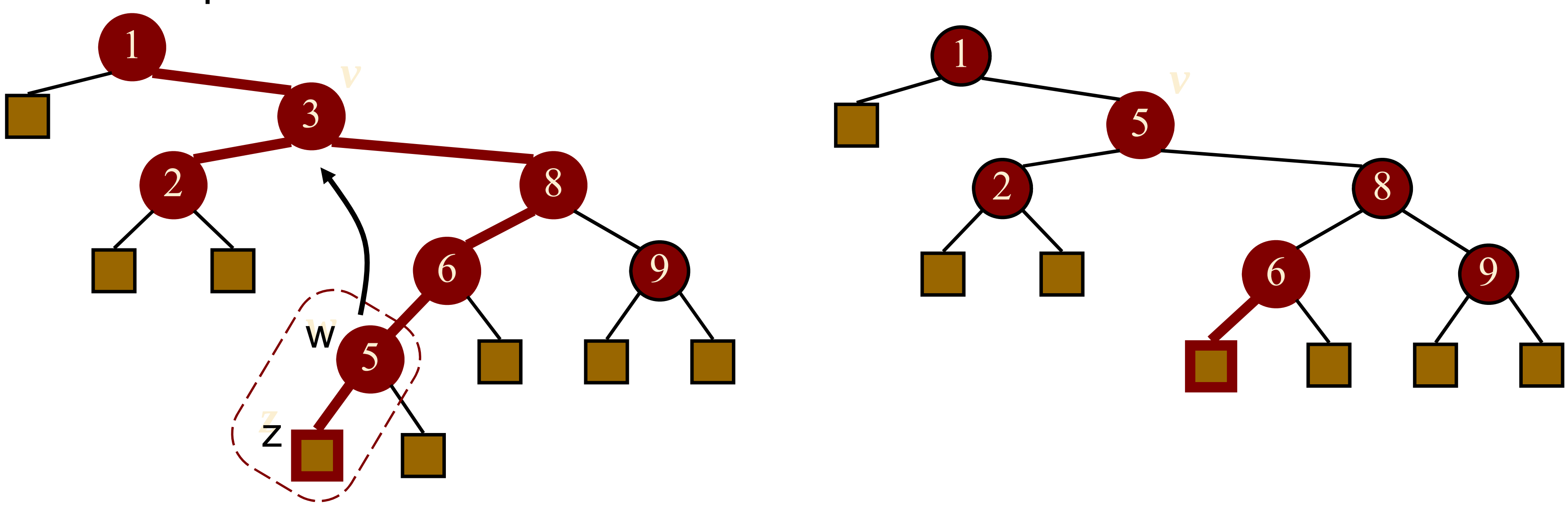
Deletion

- To perform operation remove(***k***), we search for key ***k***
- Suppose key ***k*** is in the tree, and let ***v*** be the node storing ***k***
- If node ***v*** has an external leaf child ***w***, we remove ***v*** and ***w*** from the tree with operation removeExternal(***w***), which removes ***w*** and its parent
- Example: remove 4



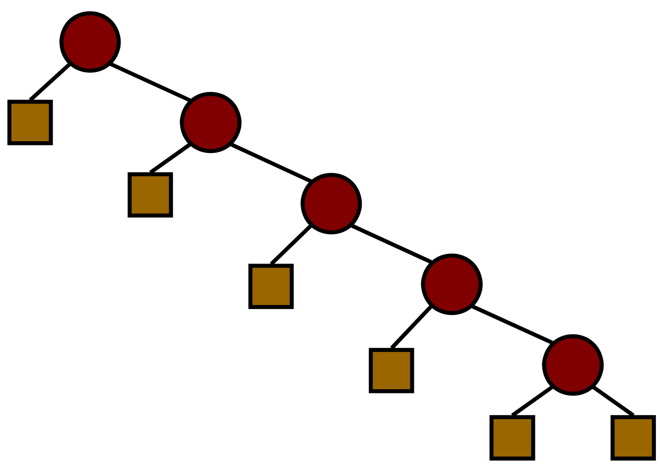
Deletion (cont.)

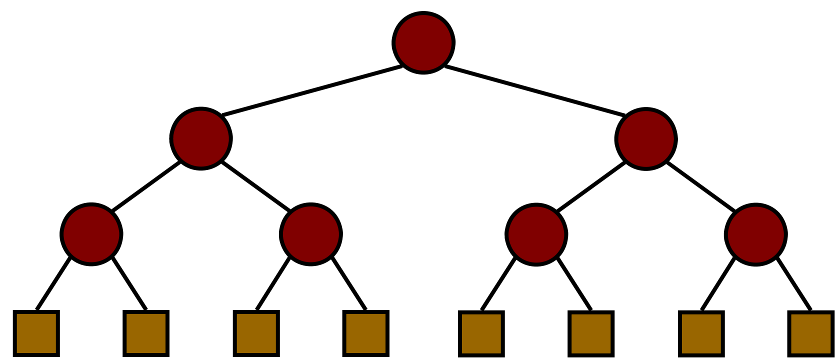
- Now consider the case where the key ***k*** to be removed is stored at a node ***v*** whose children are both internal
  - we find the internal node ***w*** that follows ***v*** in an inorder traversal
  - we copy the entry stored at ***w*** into node ***v***
  - we remove node ***w*** and its left child ***z*** (which must be a leaf) by means of operation removeExternal(***z***)
- Example: remove 3



Performance

- Consider a dictionary with n items implemented by means of a linked binary search tree of height h
  - the space used is $O(n)$
  - methods find, insert and remove take $O(h)$ time
- The height h is $O(n)$ in the worst case and $O(\log n)$ in the best case
- It is thus worthwhile to balance the tree (next topic)!





Outline

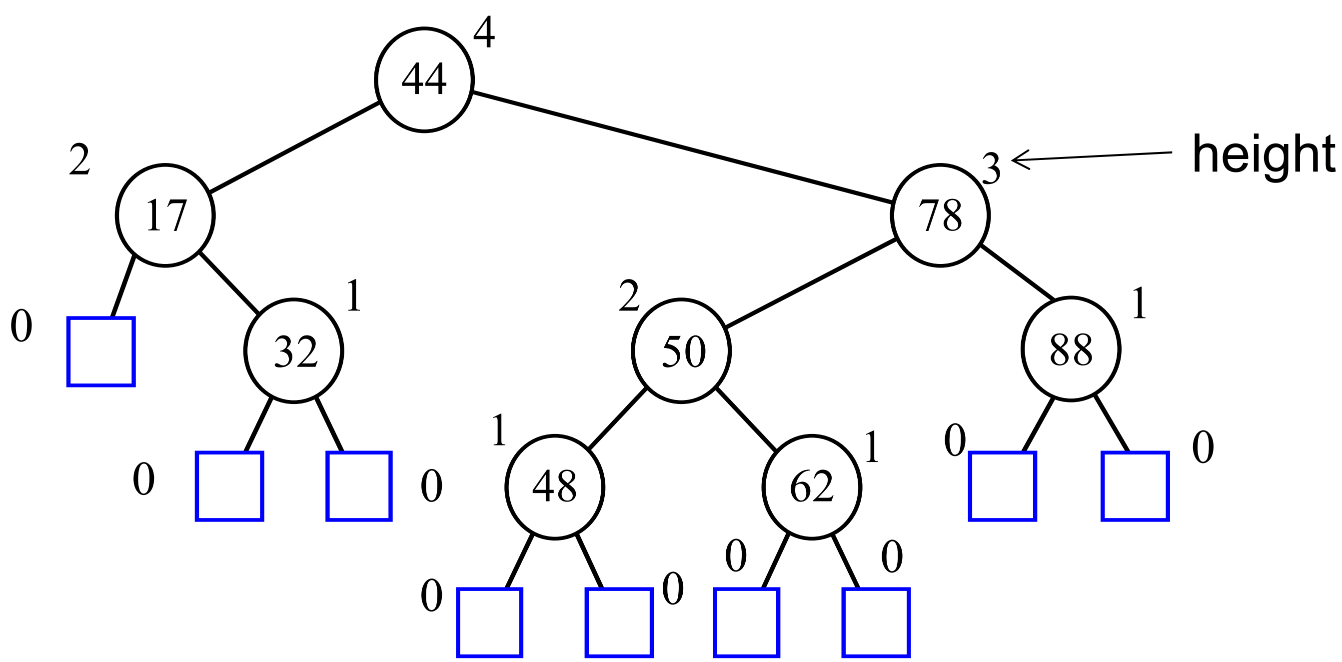
- Binary Search Trees
- **AVL Trees**
- Splay Trees

AVL Trees

- The AVL tree is the first balanced binary search tree ever invented.
- It is named after its two inventors, G.M. Adelson-Velskii and E.M. Landis, who published it in their 1962 paper "An algorithm for the organization of information."

AVL Trees

- **AVL trees are balanced.**
- An AVL Tree is a **binary search tree** in which the heights of siblings can differ by at most 1.

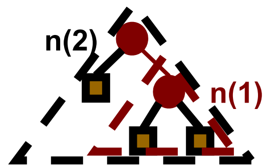


Height of an AVL Tree

- **Claim**: The ***height*** of an AVL tree storing n keys is O(log n).

Height of an AVL Tree

- **Proof**: We compute a lower bound **n(h)** on the number of internal nodes of an AVL tree of height h.
- Observe that n(1) = 1 and n(2) = 2

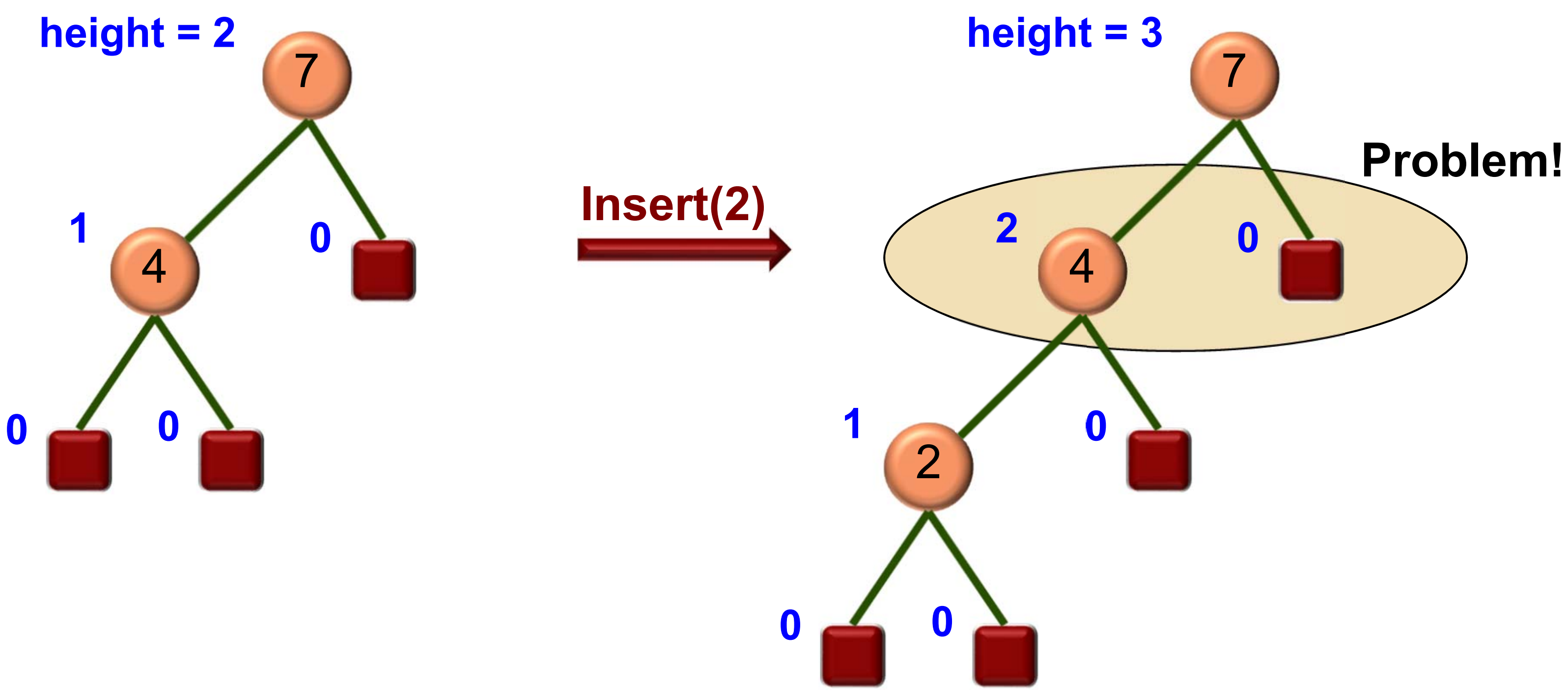


- For h > 2, a minimal AVL tree contains the root node, one minimal AVL subtree of height h - 1 and another of height h - 2.
- That is, n(h) = 1 + n(h - 1) + n(h - 2)
- Knowing n(h - 1) > n(h - 2), we get n(h) > 2n(h - 2). So

  $n(h) > 2n(h - 2)$, $n(h) > 4n(h - 4)$, $n(h) > 8n(n - 6)$, … $> 2^i n(h - 2i)$

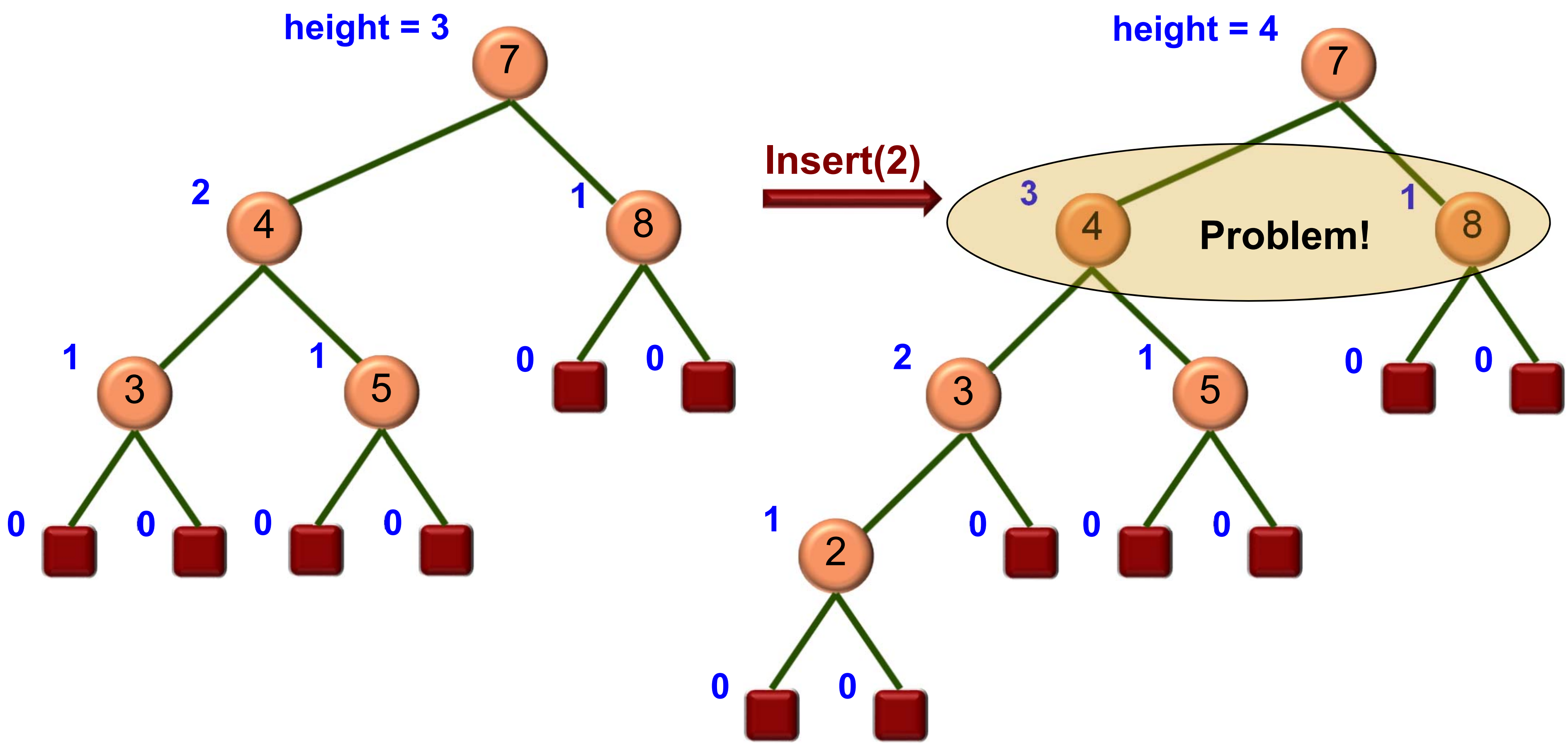
- If h is even, we let i = h/2-1, so that $n(h) > 2^{h/2-1} n(2) = 2^{h/2}$
- If h is odd, we let i = h/2-1/2, so that $n(h) > 2^{h/2-1/2} n(1) = 2^{h/2-1/2}$
- In either case, $n(h) > 2^{h/2-1}$
- Taking logarithms: h < 2log(n(h)) +2
- Thus the height of an AVL tree is O(log n)

Insertion



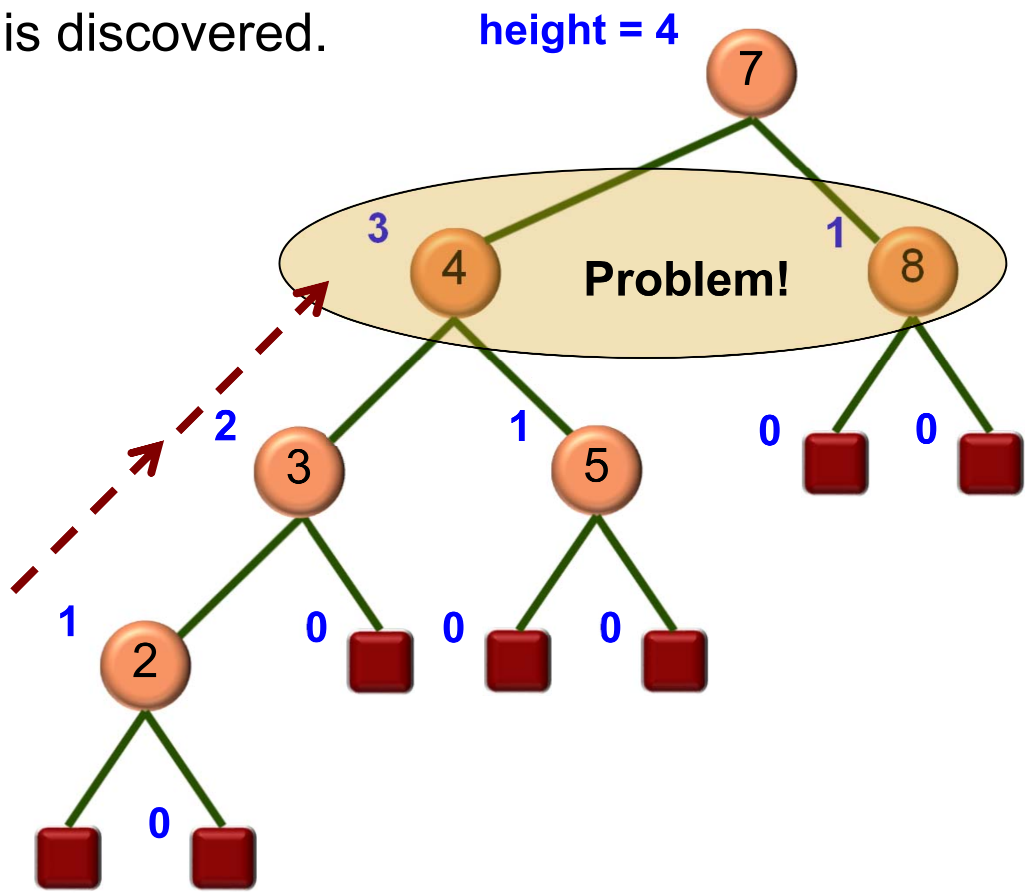
Insertion

- Imbalance may occur at any ancestor of the inserted node.



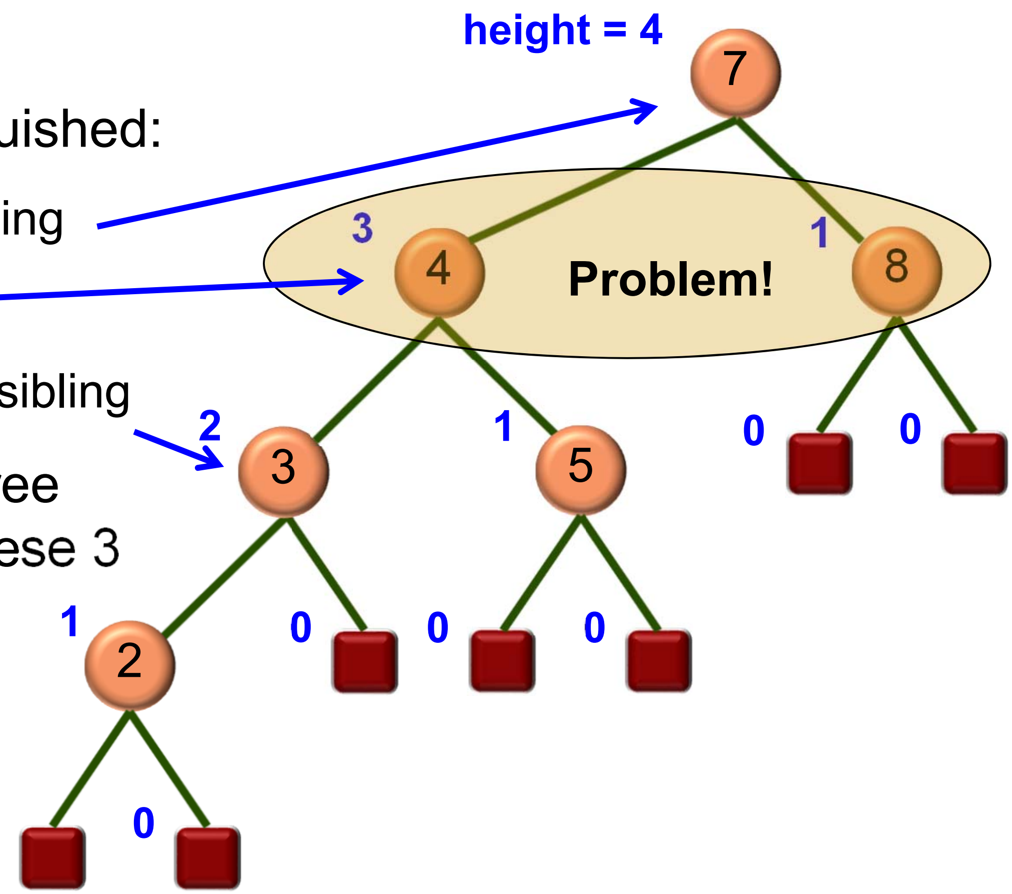
Insertion: Rebalancing Strategy

- Step 1: Search
  - Starting at the inserted node, traverse toward the root until an imbalance is discovered.



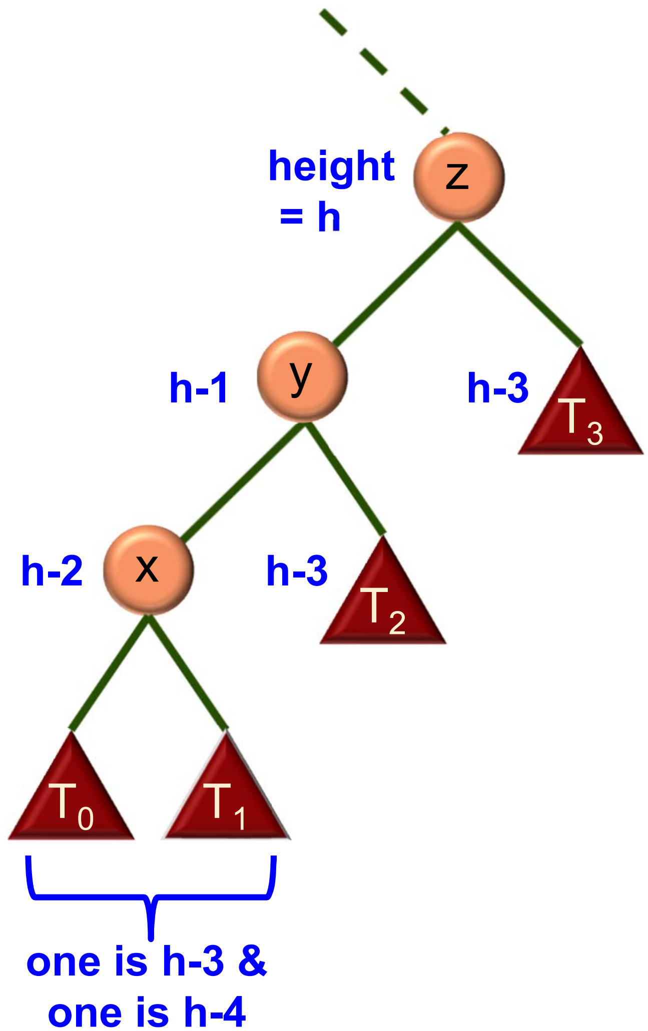
Insertion: Rebalancing Strategy

- Step 2: Repair
  - The repair strategy is called **trinode restructuring**.
  - 3 nodes x, y and z are distinguished:
    - z = the parent of the high sibling
    - y = the high sibling
    - x = the high child of the high sibling
  - We can now think of the subtree rooted at z as consisting of these 3 nodes plus their 4 subtrees



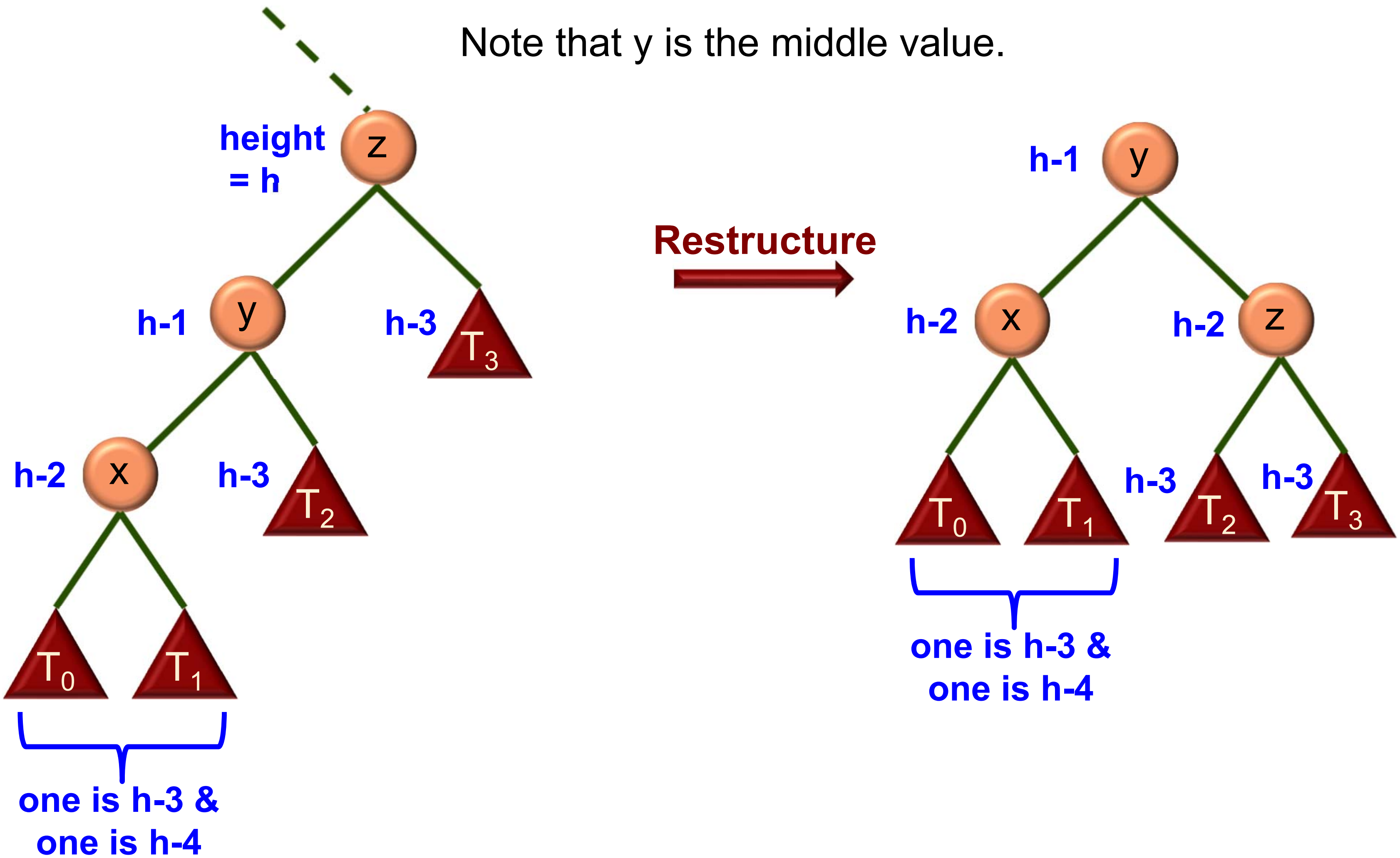
Insertion: Rebalancing Strategy

- Step 2: Repair
  - The idea is to rearrange these 3 nodes so that the middle value becomes the root and the other two becomes its children.
  - Thus the **grandparent – parent – child** structure becomes a triangular **parent – two children** structure.
  - Note that **z** must be either bigger than both **x** and **y** or smaller than both **x** and **y**.
  - Thus either **x** or **y** is made the root of this subtree.
  - Then the subtrees **T_0 – T_3** are attached at the appropriate places.
  - Since the heights of subtrees **T_0 – T_3** differ by at most 1, the resulting tree is balanced.



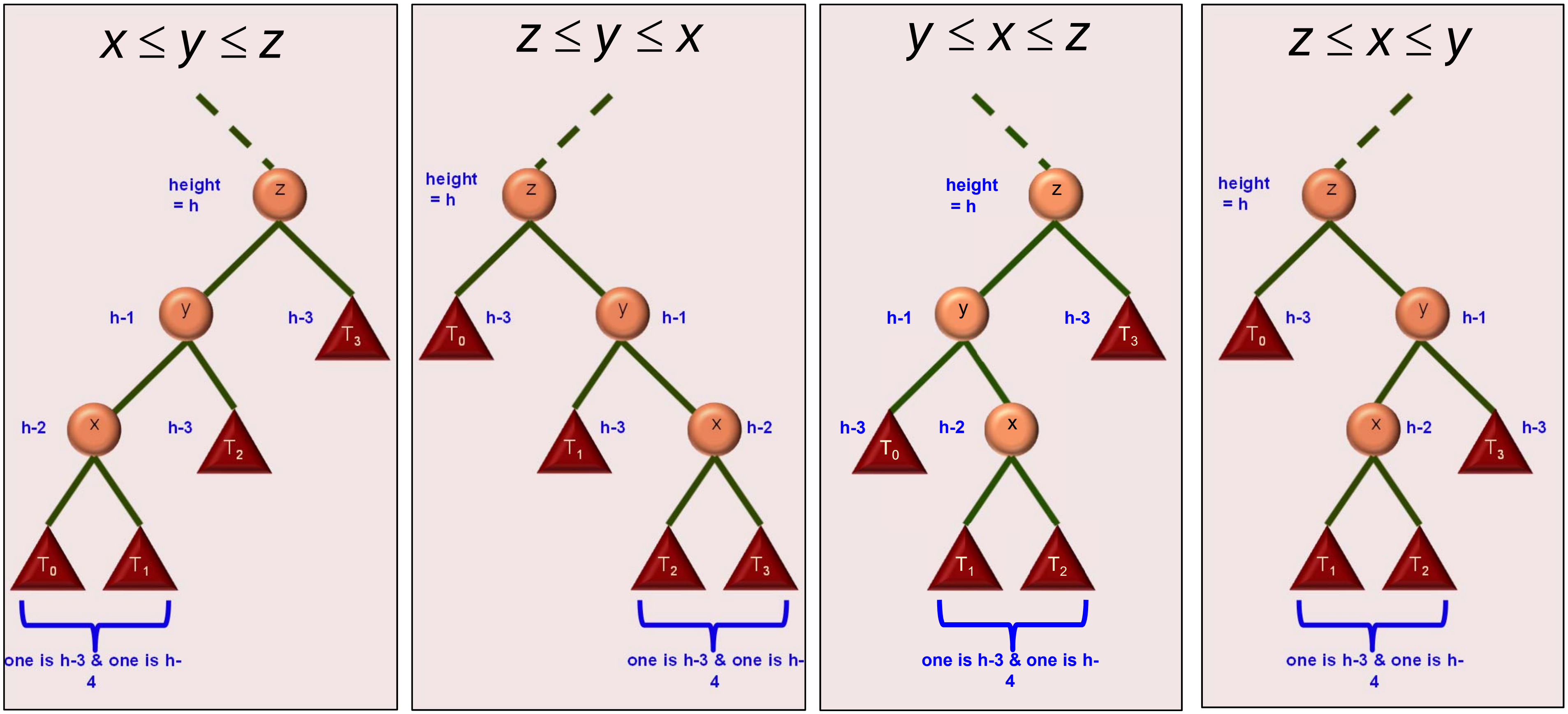
Insertion: Trinode Restructuring Example

Note that y is the middle value.



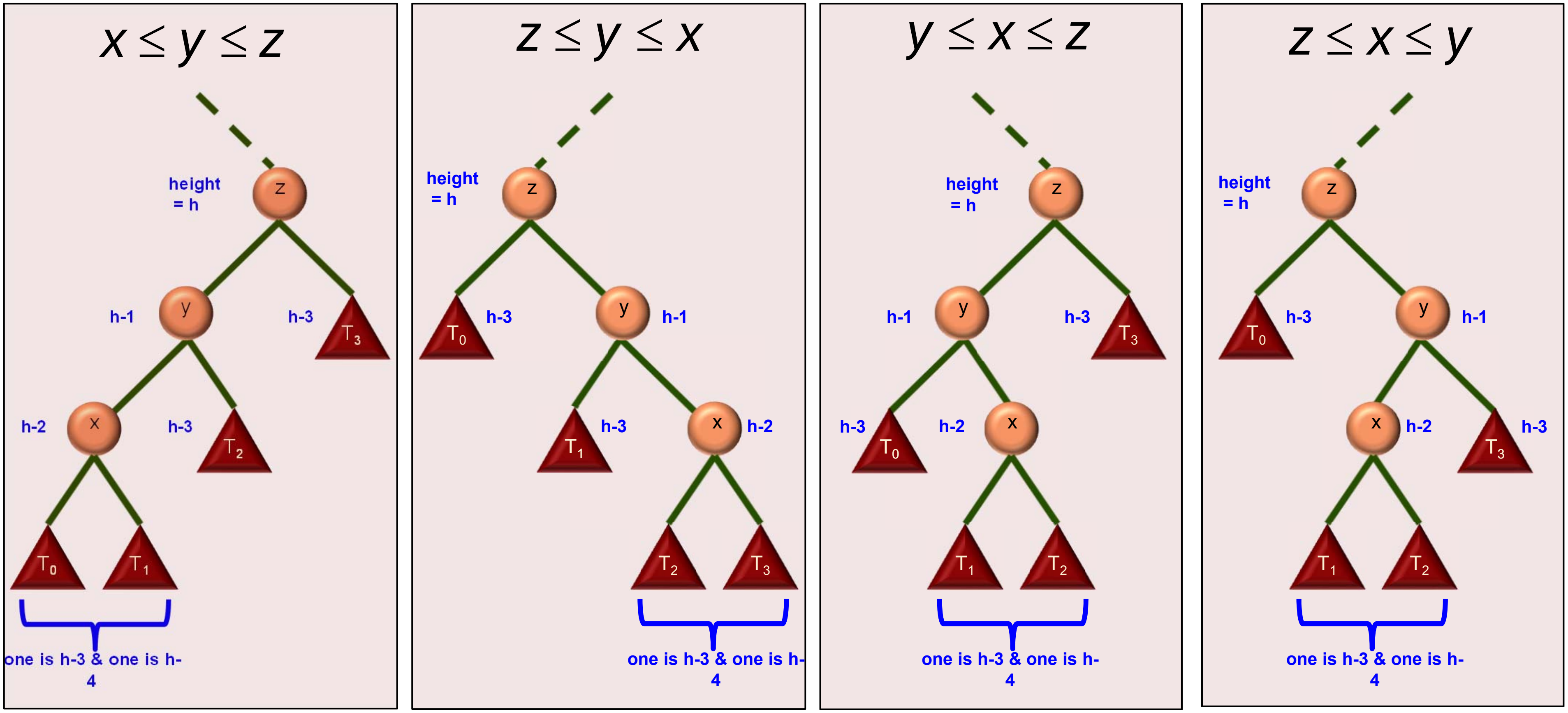
Insertion: Trinode Restructuring - 4 Cases

- There are 4 different possible relationships between the three nodes x, y and z before restructuring:



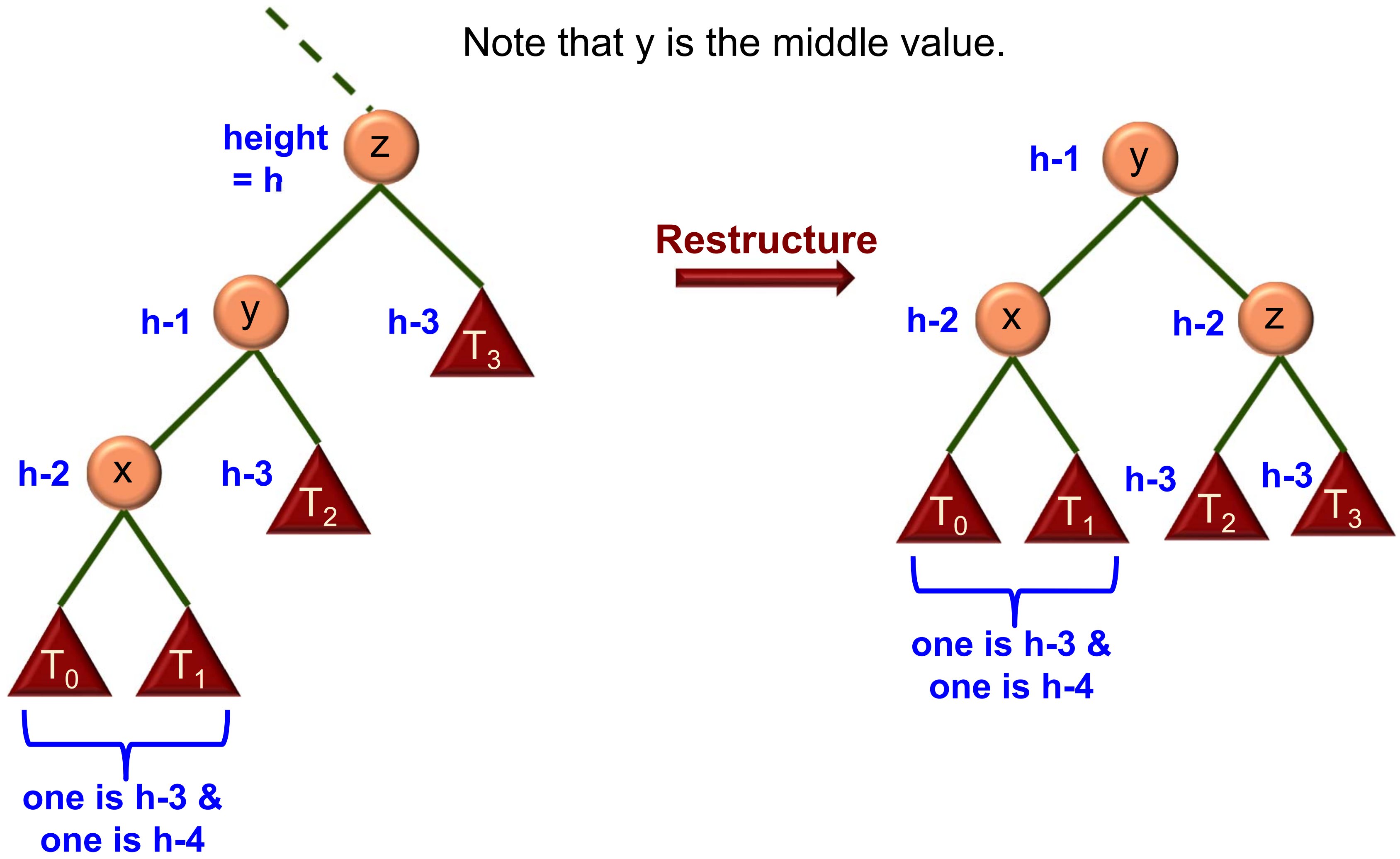
Insertion: Trinode Restructuring - 4 Cases

- This leads to 4 different solutions, all based on the same principle.

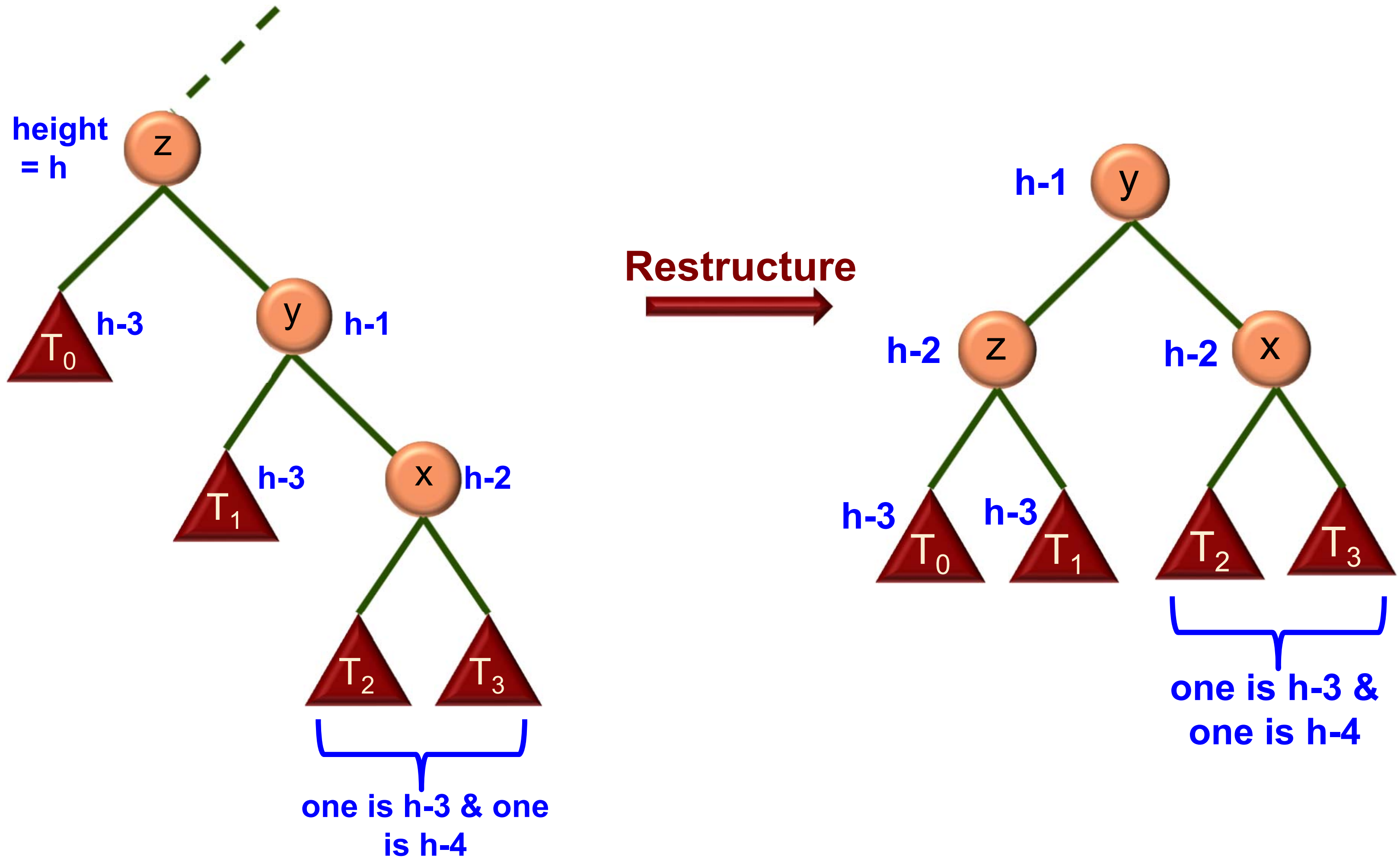


Insertion: Trinode Restructuring - Case 1

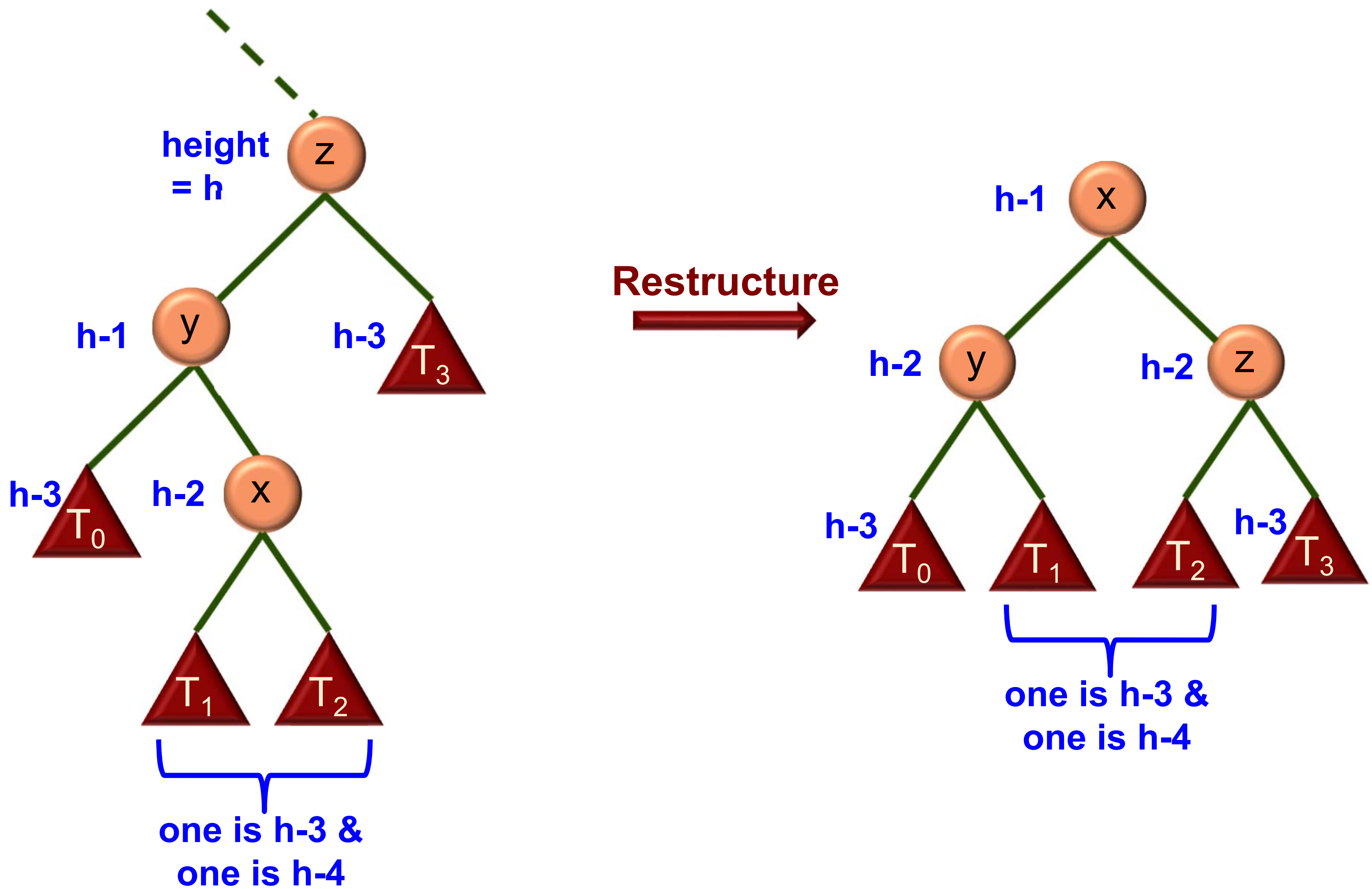
Note that y is the middle value.



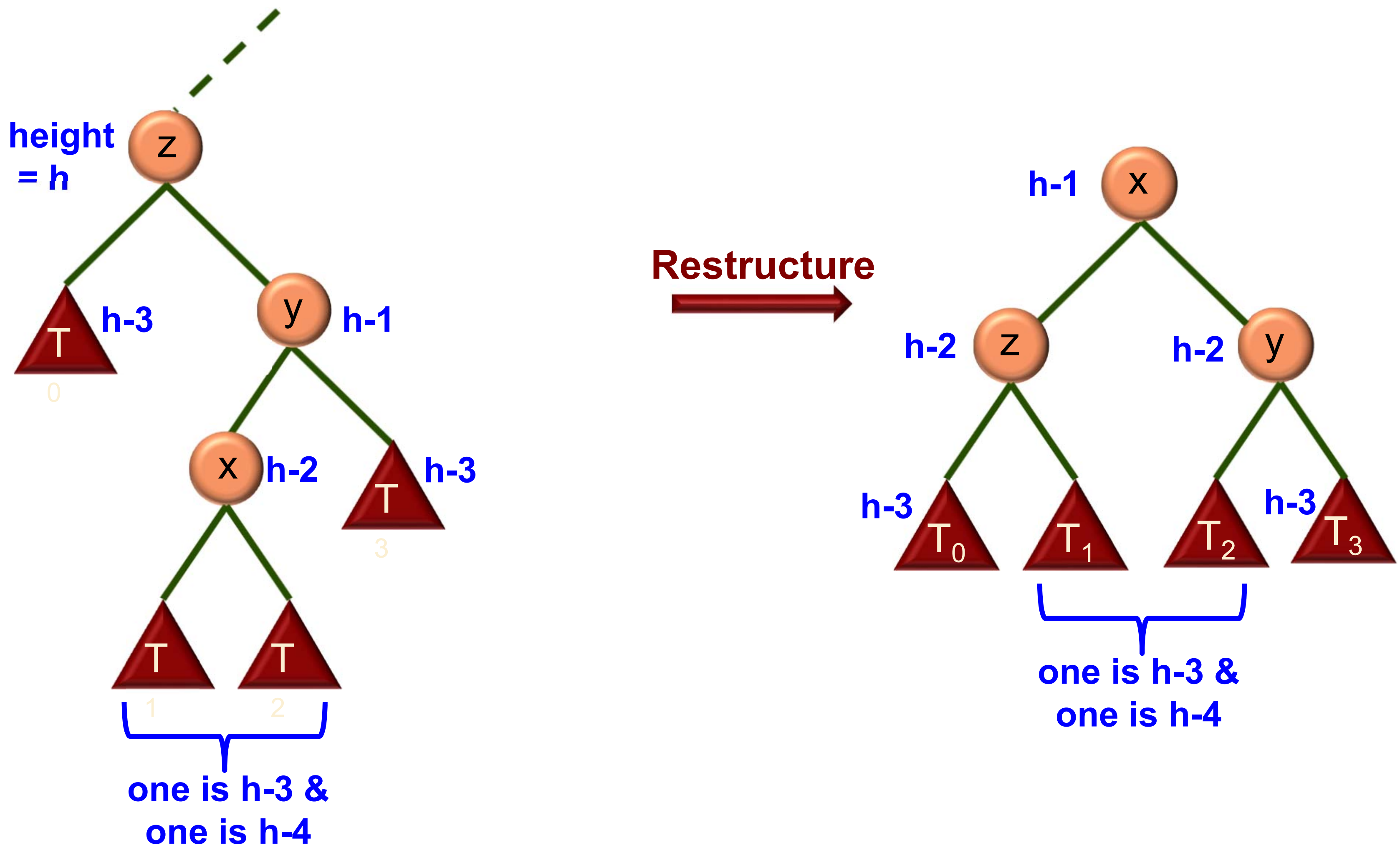
Insertion: Trinode Restructuring - Case 2



Insertion: Trinode Restructuring - Case 3

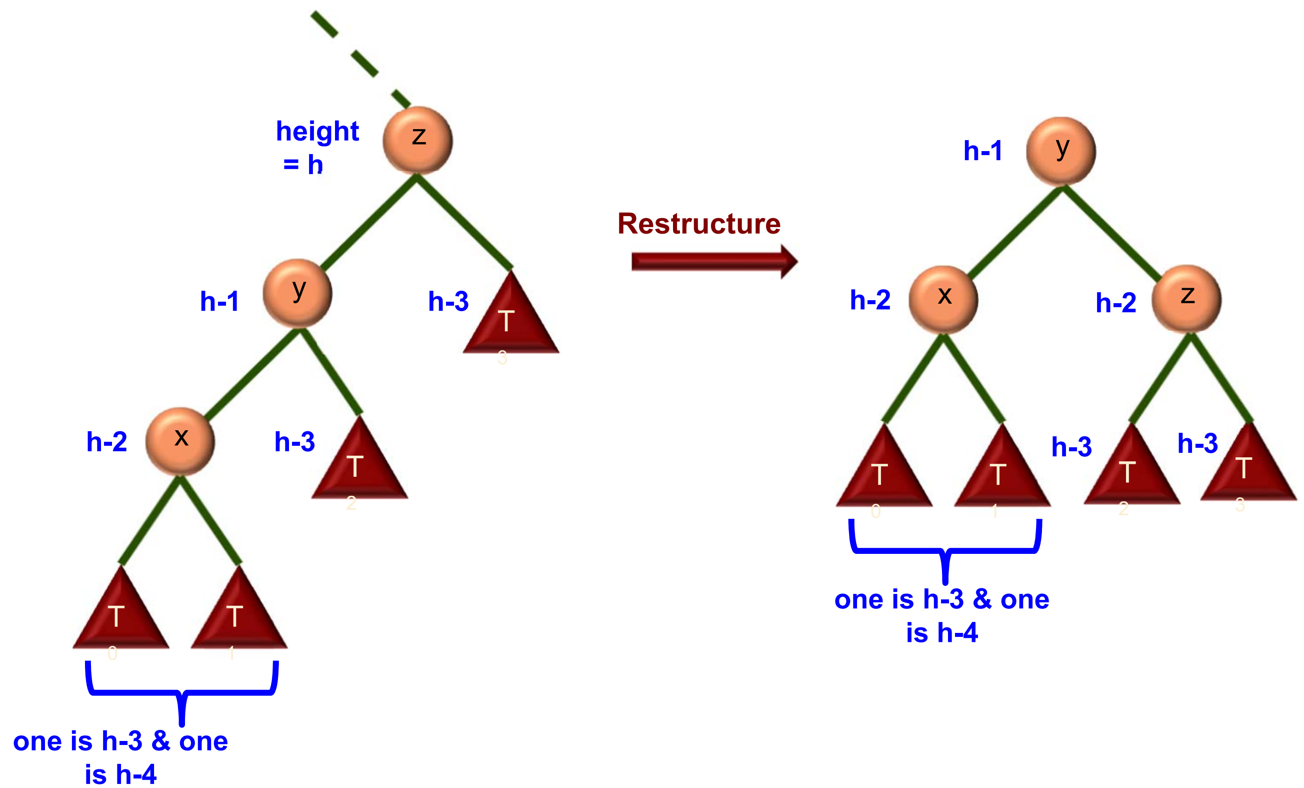


Insertion: Trinode Restructuring - Case 4



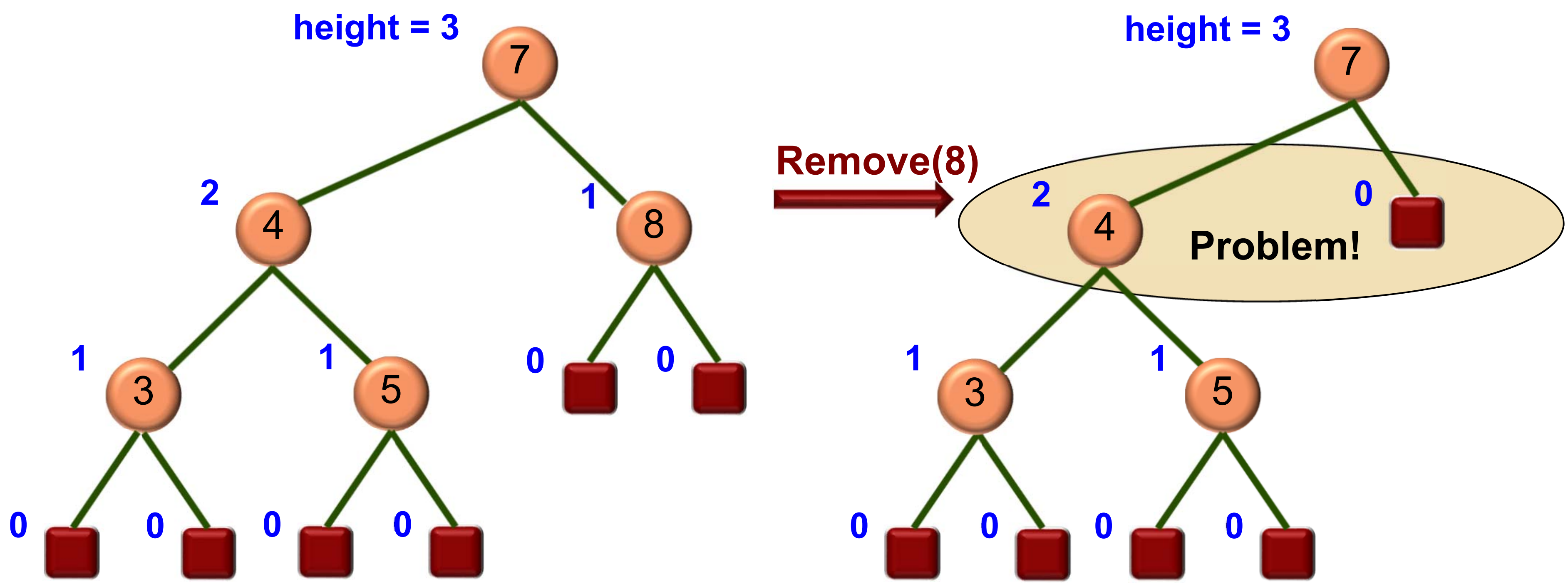
Insertion: Trinode Restructuring - The Whole Tree

- Do we have to repeat this process further up the tree?
- No!
  - The tree was balanced before the insertion.
  - Insertion raised the height of the subtree by 1.
  - Rebalancing lowered the height of the subtree by 1.
  - Thus the whole tree is still balanced.



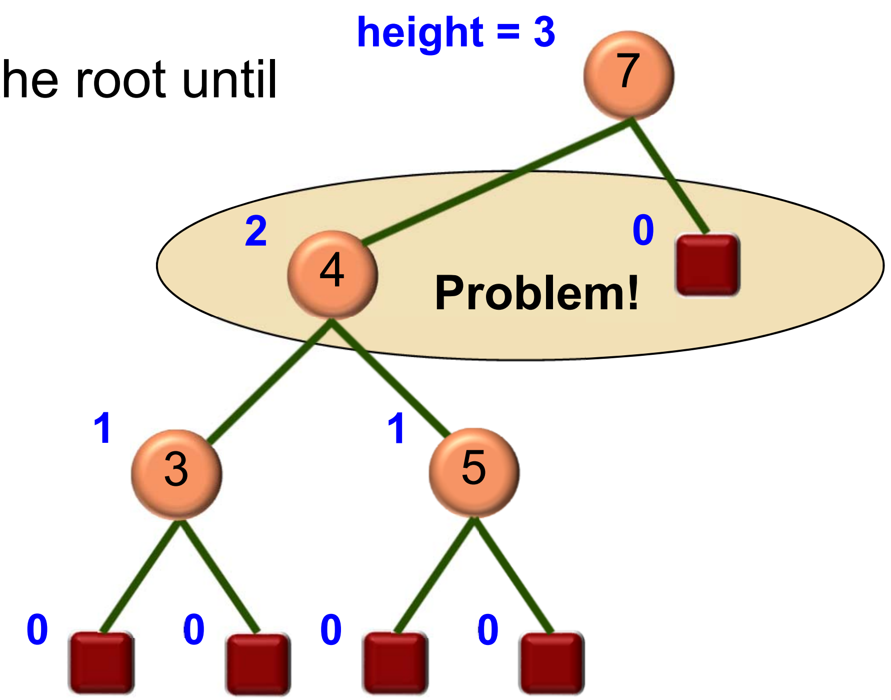
Removal

- Imbalance may occur at an ancestor of the removed node.



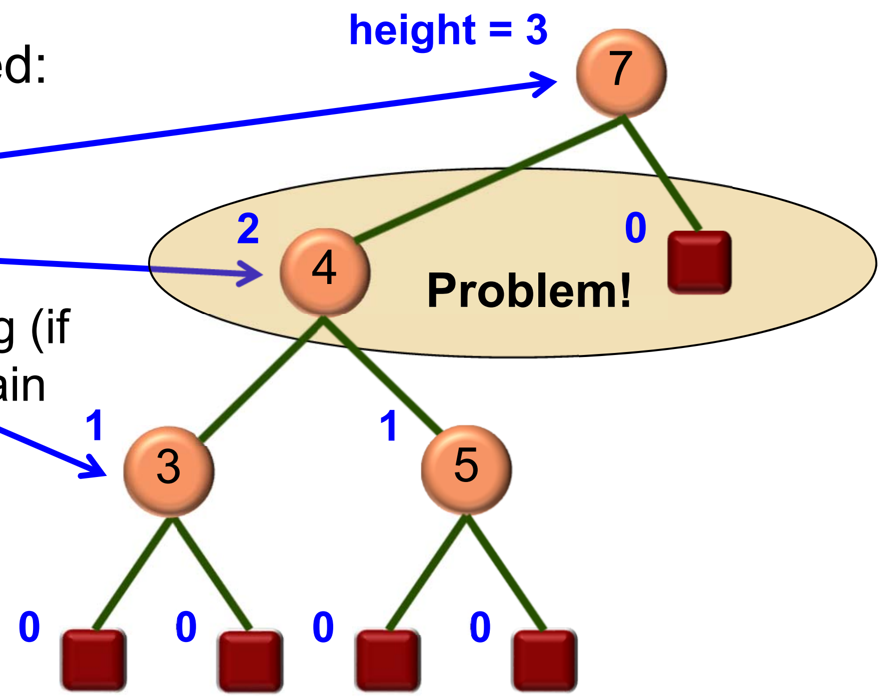
Removal: Rebalancing Strategy

- Step 1: Search
  - Let ***w*** be the node actually removed (i.e., the node matching the key if it has a leaf child, otherwise the node following in an in-order traversal.
  - Starting at ***w***, traverse toward the root until an imbalance is discovered.



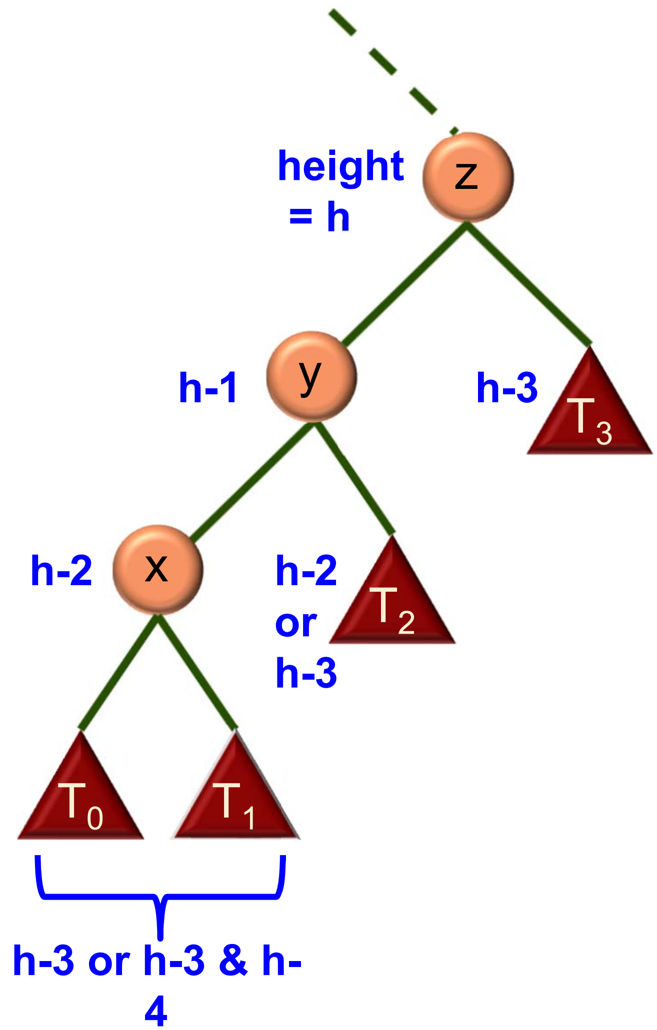
Removal: Rebalancing Strategy

- Step 2: Repair
  - We again use **trinode restructuring**.
  - 3 nodes x, y and z are distinguished:
    - z = the parent of the high sibling
    - y = the high sibling
    - x = the high child of the high sibling (if children are equally high, keep chain linear)



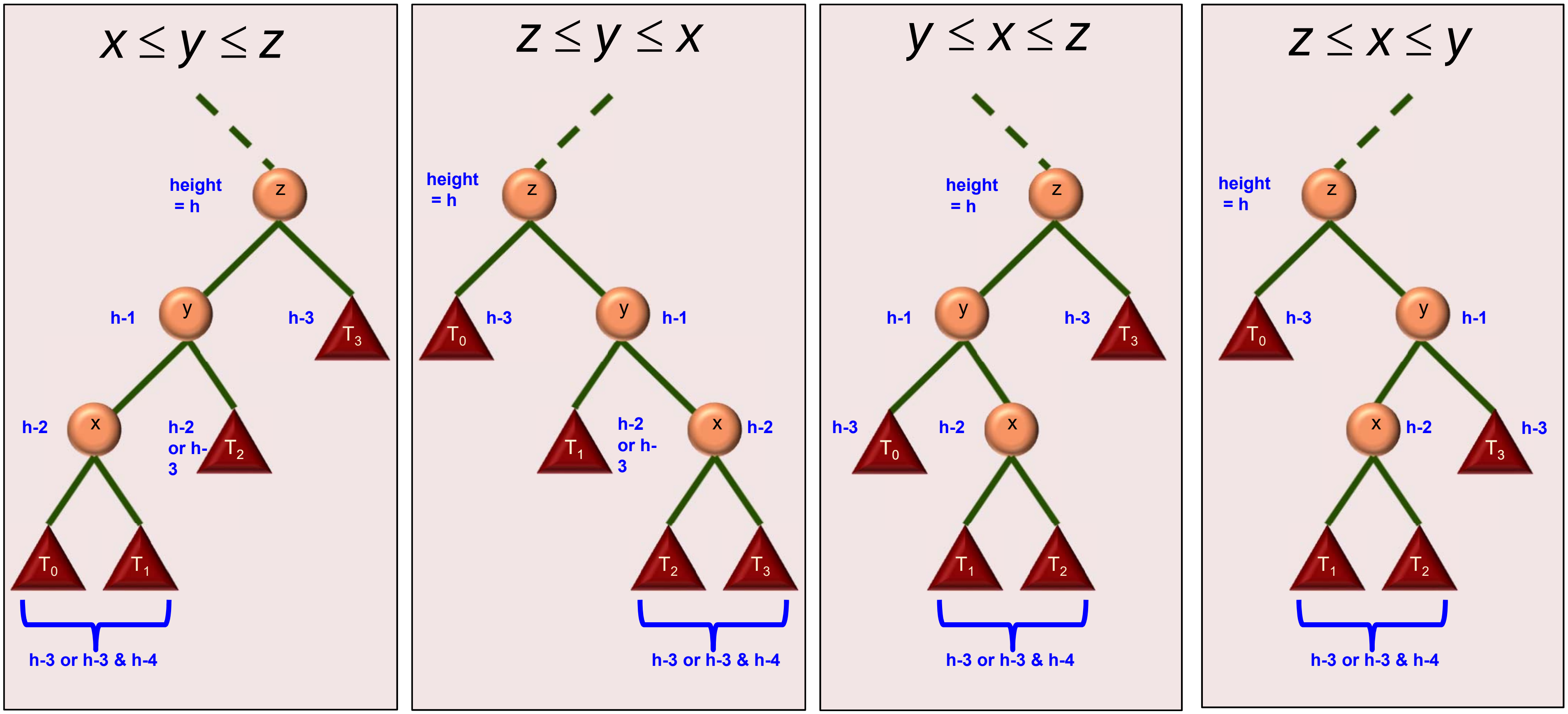
Removal: Rebalancing Strategy

- Step 2: Repair
  - The idea is to rearrange these 3 nodes so that the middle value becomes the root and the other two becomes its children.
  - Thus the **grandparent – parent – child** structure becomes a triangular **parent – two children** structure.
  - Note that **z** must be either bigger than both **x** and **y** or smaller than both **x** and **y**.
  - Thus either **x** or **y** is made the root of this subtree, and **z** is lowered by 1.
  - Then the subtrees $\mathbf{T_0 – T_3}$ are attached at the appropriate places.
  - Although the subtrees $\mathbf{T_0 – T_3}$ can differ in height by up to 2, after restructuring, sibling subtrees will differ by at most 1.



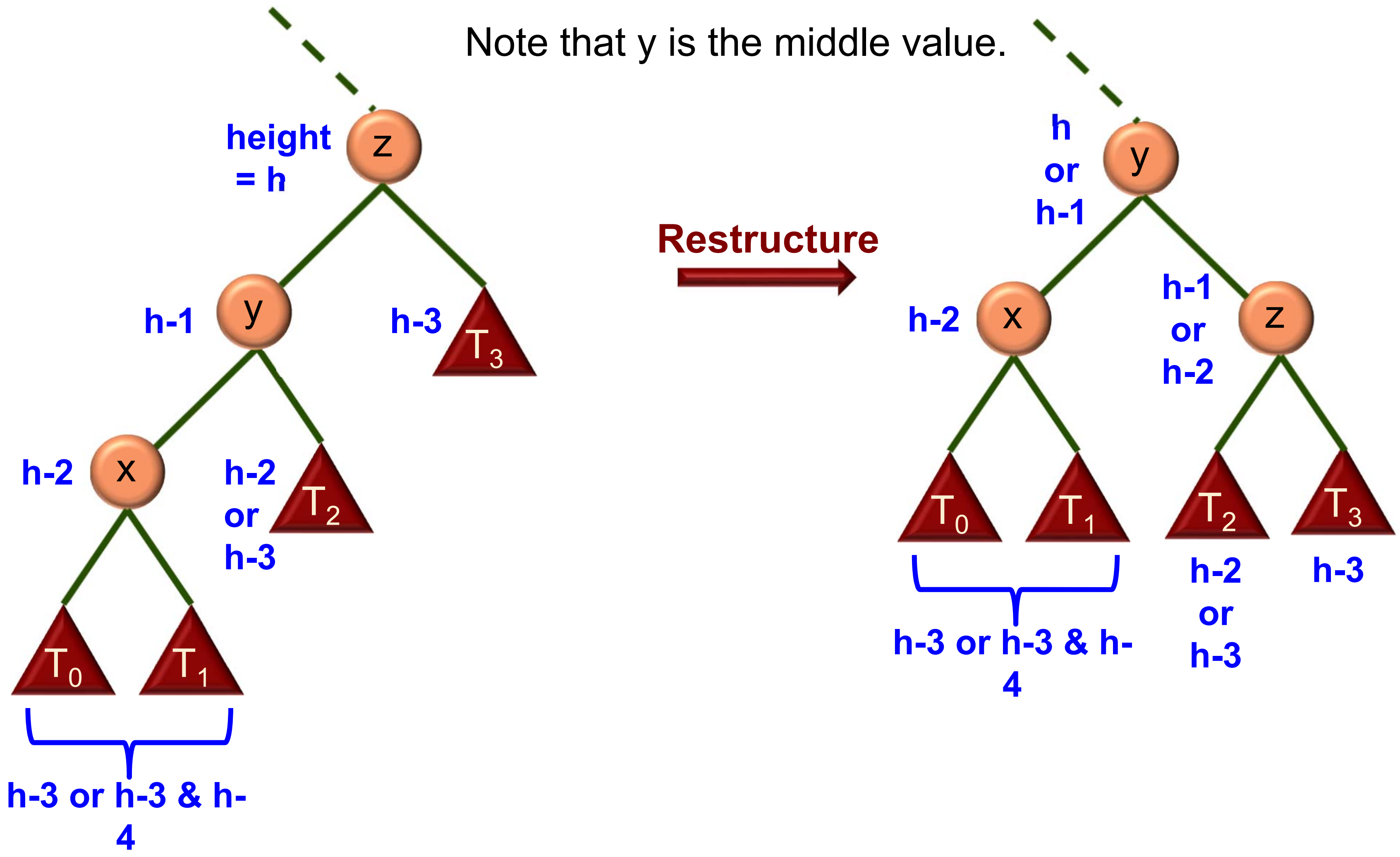
Removal: Trinode Restructuring - 4 Cases

- There are 4 different possible relationships between the three nodes x, y and z before restructuring:

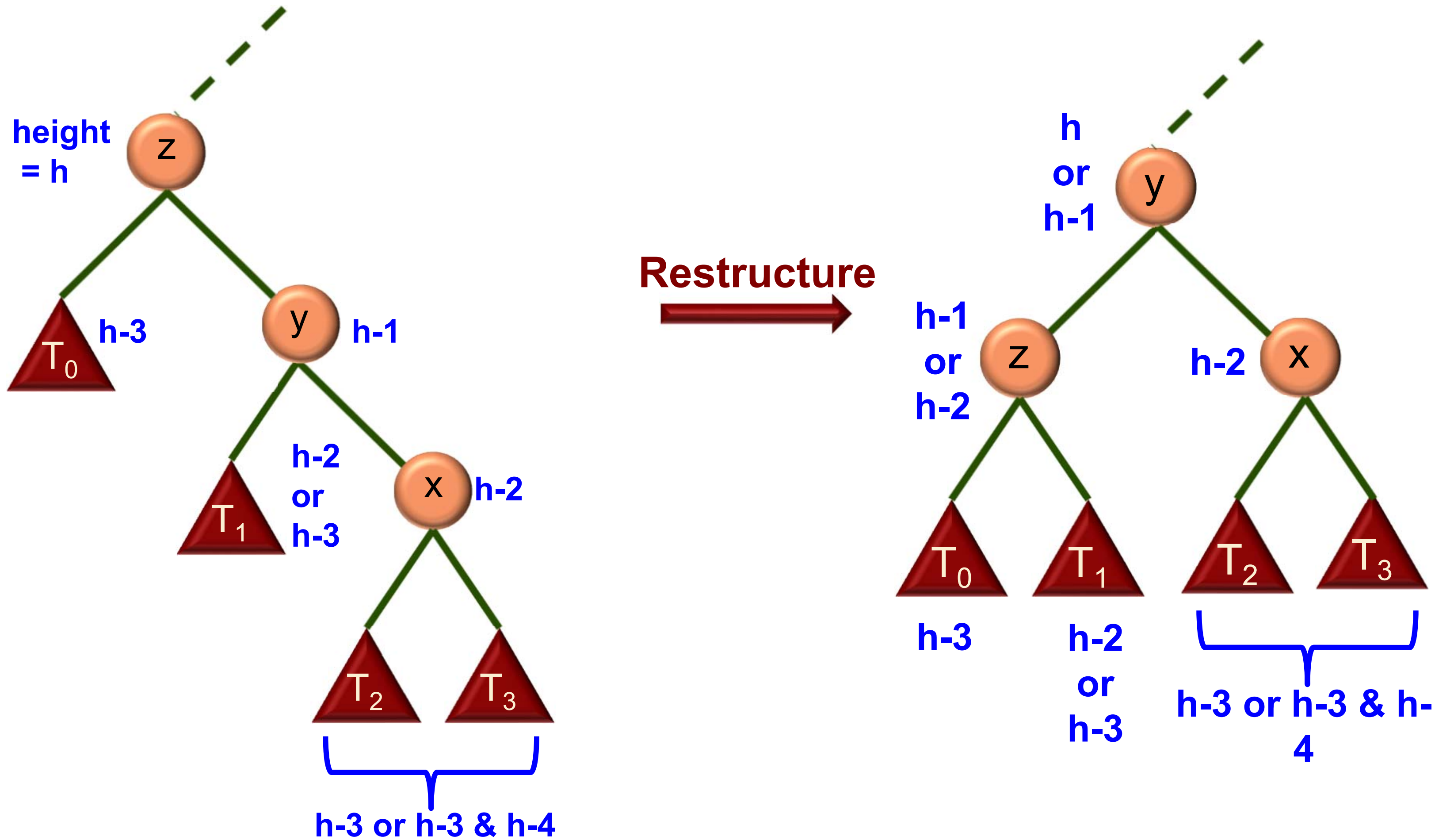


Removal: Trinode Restructuring - Case 1

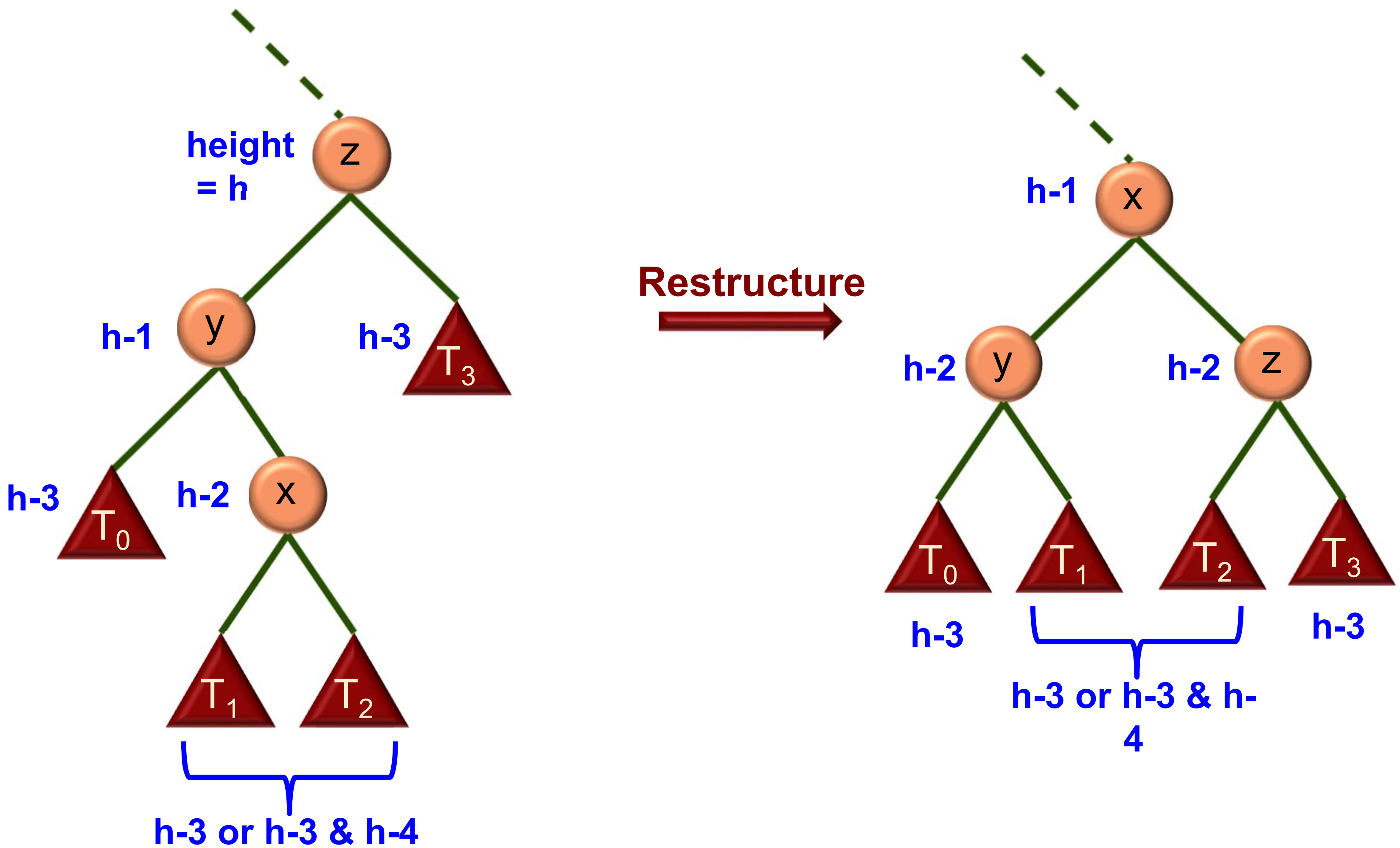
Note that y is the middle value.



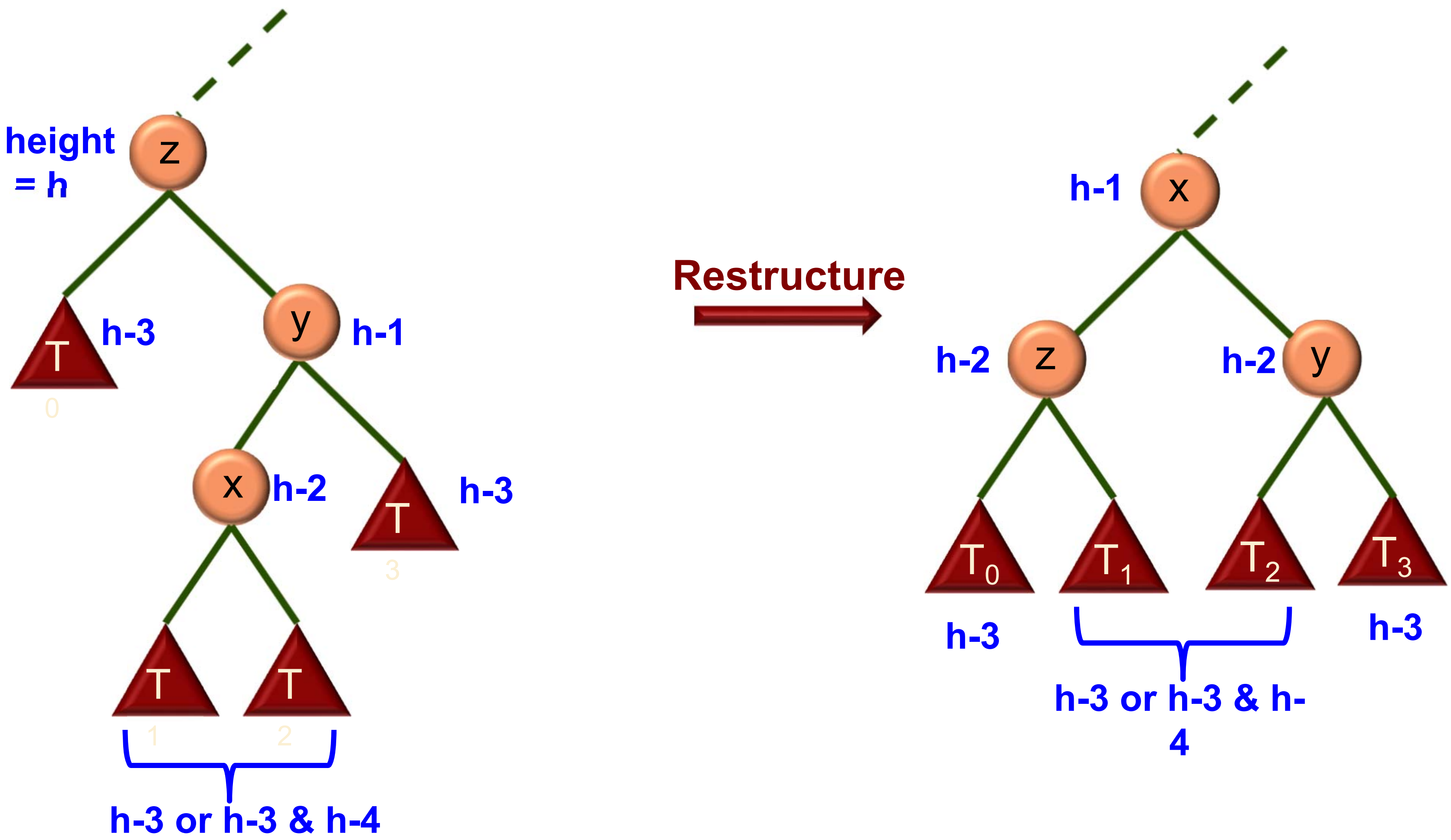
Removal: Trinode Restructuring - Case 2



Removal: Trinode Restructuring - Case 3



Removal: Trinode Restructuring - Case 4



Removal: Rebalancing Strategy

- Step 2: Repair
  - Unfortunately, trinode restructuring may reduce the height of the subtree, causing another imbalance further up the tree.
  - Thus this search and repair process must in the worst case be repeated until we reach the root.

Java Implementation of AVL Trees

- Please see text

Running Times for AVL Trees

- a single restructure is O(1)
  - using a linked-structure binary tree
- find is O(log n)
  - height of tree is O(log n), no restructures needed
- insert is O(log n)
  - initial find is O(log n)
  - Restructuring is O(1)
- remove is O(log n)
  - initial find is O(log n)
  - Restructuring up the tree, maintaining heights is O(log n)

AVLTree Example

Outline

- Binary Search Trees
- AVL Trees
- **Splay Trees**

Splay Trees

- Self-balancing BST
- Invented by Daniel Sleator and Bob Tarjan
- Allows quick access to recently accessed elements
- Bad: worst-case O(n)
- Good: average (amortized) case O(log n)
- Often perform better than other BSTs in practice

D. Sleator

R. Tarjan

Splaying

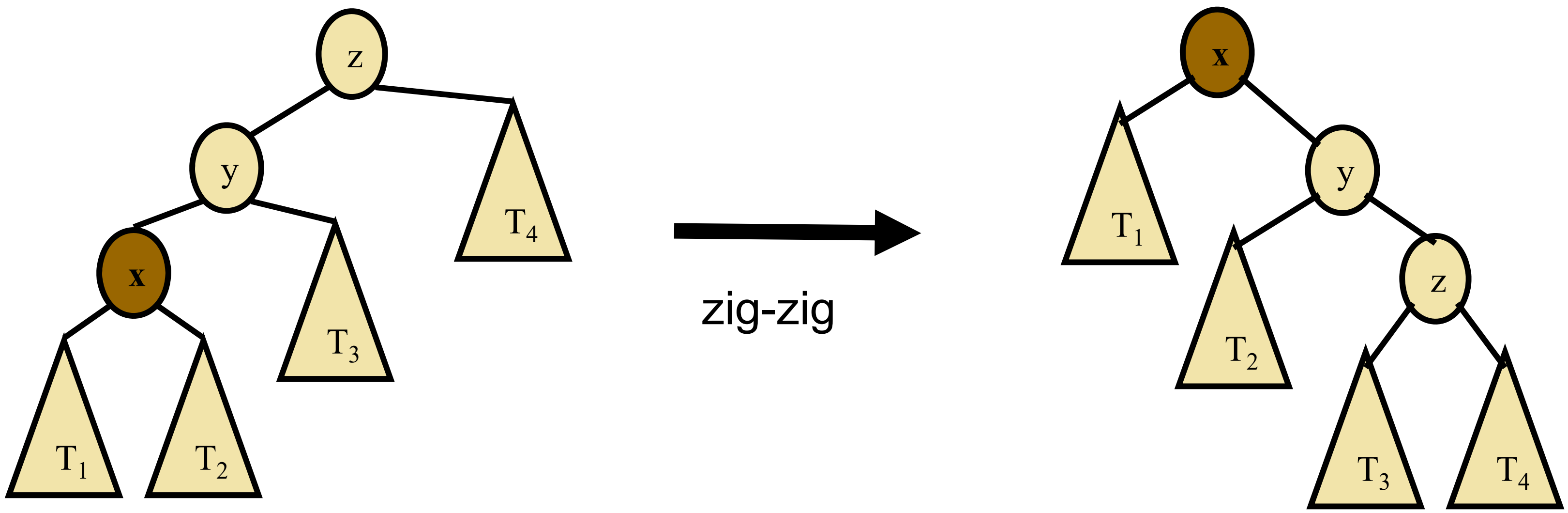
- Splaying is an operation performed on a node that iteratively moves the node to the root of the tree.
- In splay trees, each BST operation (find, insert, remove) is augmented with a splay operation.
- In this way, recently searched and inserted elements are near the top of the tree, for quick access.

3 Types of Splay Steps

- Each splay operation on a node consists of a sequence of splay steps.
- Each splay step moves the node up toward the root by 1 or 2 levels.
- There are 2 types of step:
  - Zig-Zig
  - Zig-Zag
  - Zig
- These steps are iterated until the node is moved to the root.

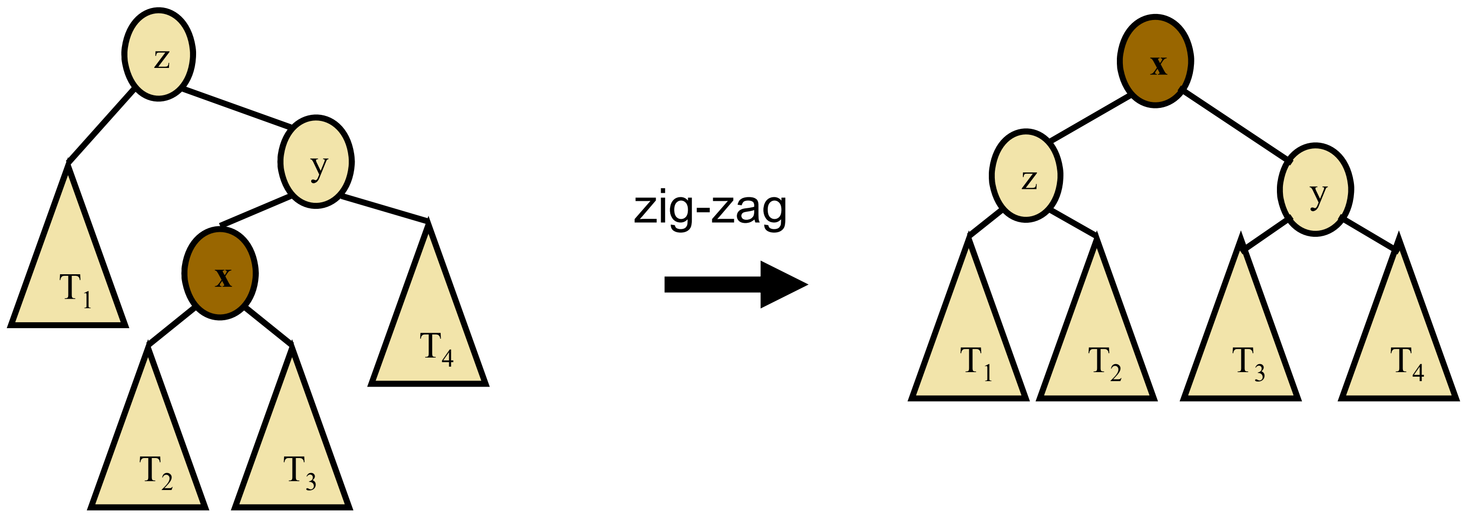
Zig-Zig

- Performed when the node x forms a linear chain with its parent and grandparent.
  - i.e., right-right or left-left



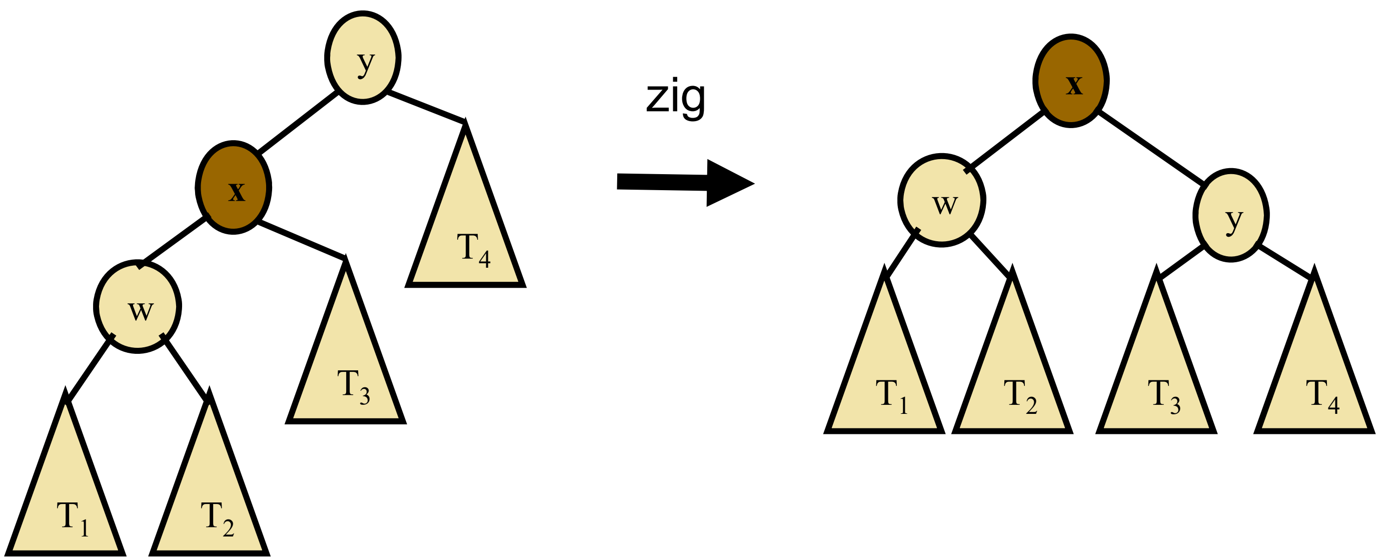
Zig-Zag

- Performed when the node x forms a non-linear chain with its parent and grandparent
  - i.e., right-left or left-right

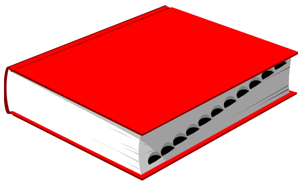


Zig

- Performed when the node x has no grandparent
  - i.e., its parent is the root



Splay Trees & Ordered Dictionaries

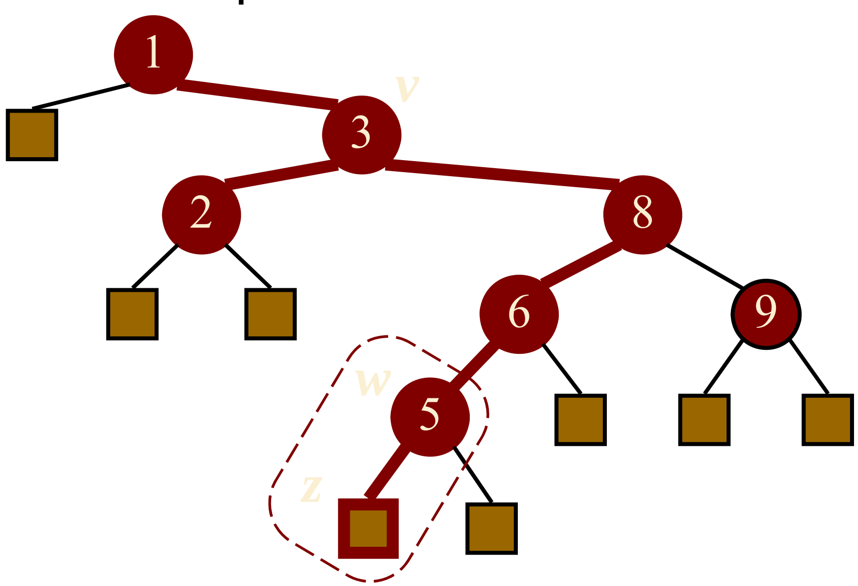


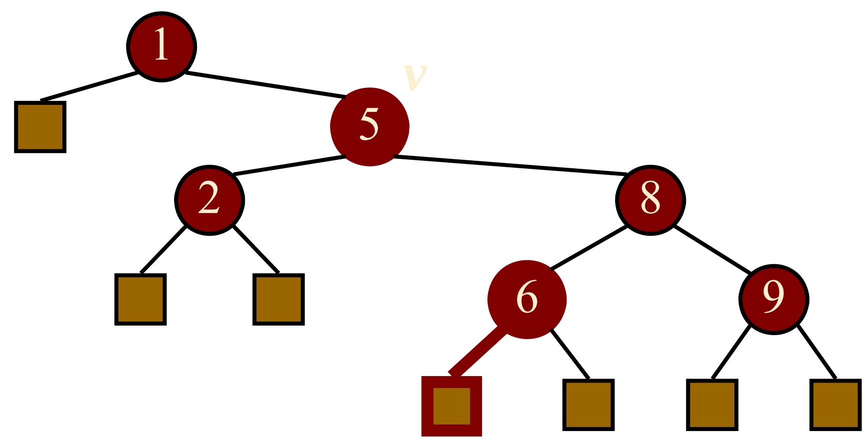
- which nodes are splayed after each operation?

method	splay node
find(k)	if key found, use that node if key not found, use parent of external node where search terminated
insert(k,v)	use the new node containing the entry inserted
remove(k)	use the parent of the internal node ***w*** that was actually removed from the tree (the parent of the node that the removed item was swapped with)

Recall BST Deletion

- Now consider the case where the key k to be removed is stored at a node v whose children are both internal
  - we find the internal node w that follows v in an inorder traversal
  - we copy $key(w)$ into node v
  - we remove node w and its left child z (which must be a leaf) by means of operation removeExternal(z)
- Example: remove 3 – which node will be splayed?





Note on Deletion

- The text (Goodrich, p. 463) uses a different convention for BST deletion in their splaying example
  - Instead of deleting the leftmost internal node of the right subtree, they delete the rightmost internal node of the left subtree.
  - We will stick with the convention of deleting the leftmost internal node of the right subtree (the node immediately following the element to be removed in an inorder traversal).

Splay Tree Example

Performance

- Worst-case is O(n)
  - Example:
    - Find all elements in sorted order
    - This will make the tree a left linear chain of height n, with the smallest element at the bottom
    - Subsequent search for the smallest element will be O(n)

Performance

- Average-case is O(log n)
  - Proof uses amortized analysis
  - We will not cover this
- Operations on more frequently-accessed entries are faster.
  - Given a sequence of *m* operations, the running time to access entry *i* is:

    $$O\left(\log\left(m / f(i)\right)\right)$$

    where *f(i)* is the number of times entry *i* is accessed.

Other Forms of Search Trees

- (2, 4) Trees
  - These are multi-way search trees (not binary trees) in which internal nodes have between 2 and 4 children
  - Have the property that all external nodes have exactly the same depth.
  - Worst-case O(log n) operations
  - Somewhat complicated to implement
- Red-Black Trees
  - Binary search trees
  - Worst-case O(log n) operations
  - Somewhat easier to implement
  - Requires only O(1) structural changes per update

Summary

- Binary Search Trees
- AVL Trees
- Splay Trees