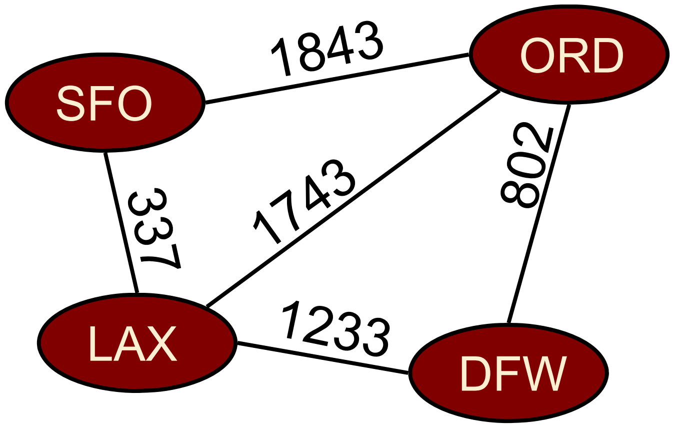
Graphs – Breadth First Search



Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness

Outline

- **BFS Algorithm**
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness

Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G
- BFS on a graph with $|V|$ vertices and $|E|$ edges takes $O(|V|+|E|)$ time
- BFS can be further extended to solve other graph problems
  - Cycle detection
  - **Find and report a path with the minimum number of edges between two given vertices**

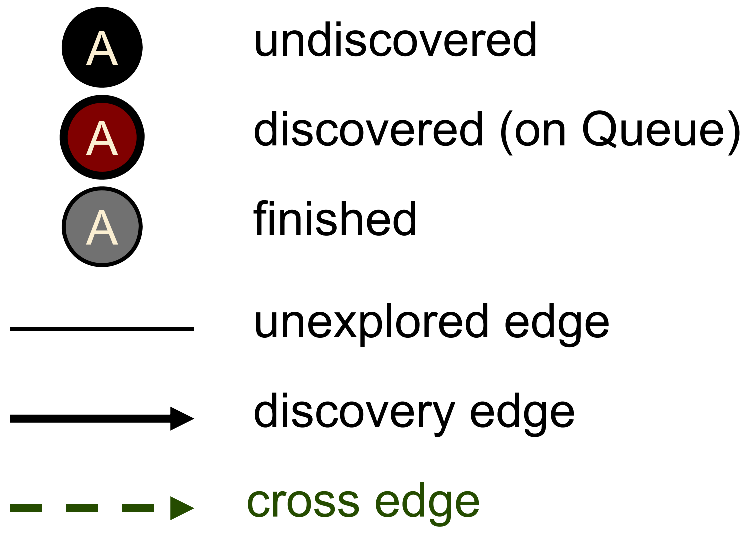
BFS Algorithm Pattern

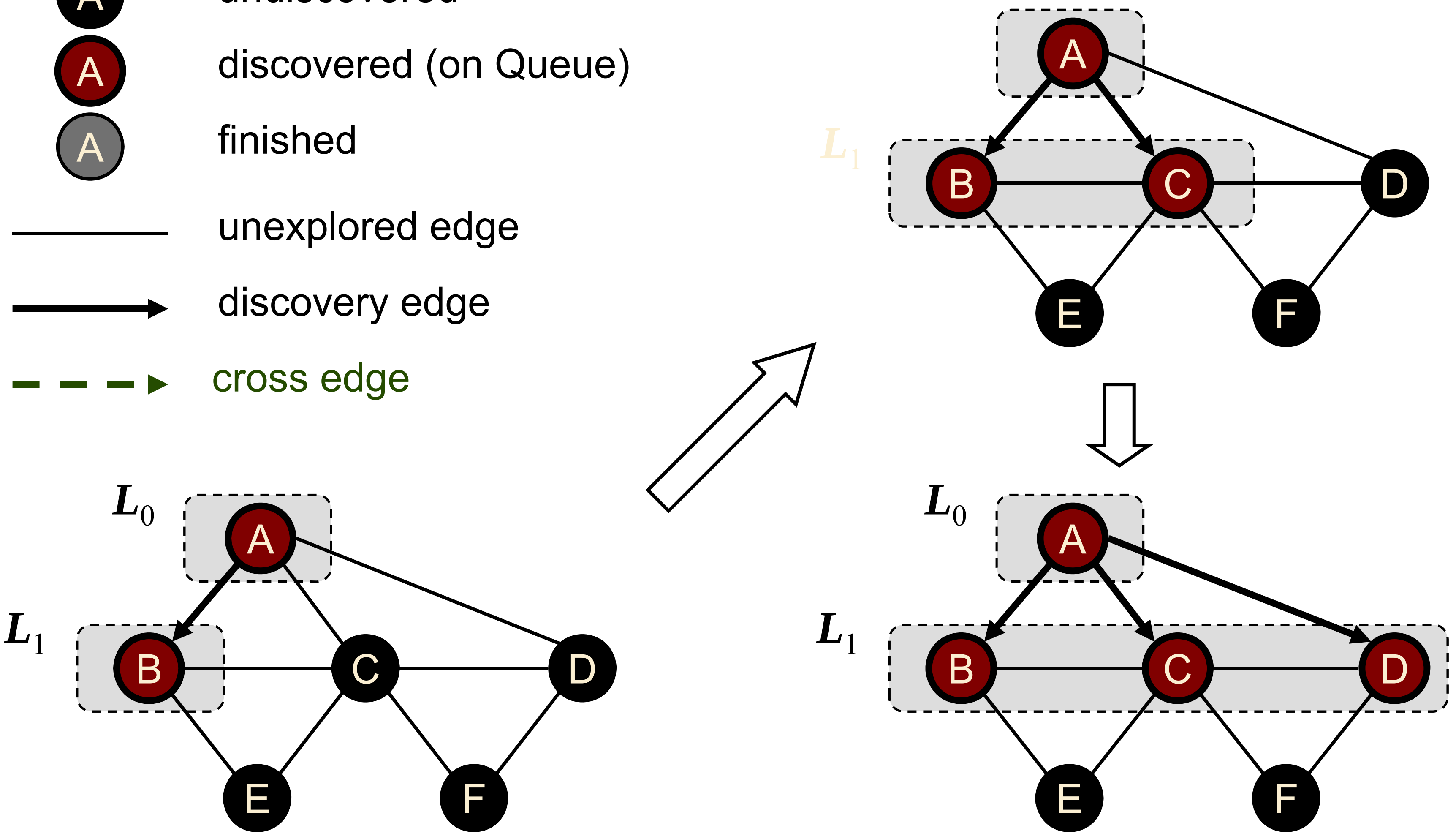
```
BFS(G,s)
Precondition: G is a graph, s is a vertex in G
Postcondition: all vertices in G reachable from s have been visited
    for each vertex u $\in V[G]$
        color[u] $\leftarrow$ BLACK //initialize vertex
    colour[s] $\leftarrow$ RED
    Q.enqueue($s$)
    while Q $\neq \varnothing$
        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
            if color[$v$] = BLACK
                colour[v] $\leftarrow$ RED
                Q.enqueue($v$)
        $colour[u] \leftarrow GRAY$
```

BFS is a Level-Order Traversal

- Notice that in BFS exploration takes place on a wavefront consisting of nodes that are all the same distance from the source *s*.
- We can label these successive wavefronts by their distance: L_0, L_1, …

BFS Example


A
undiscovered
A
discovered (on Queue)
A
finished
unexplored edge
discovery edge
cross edge


L_0
L_1
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L_0
L_1
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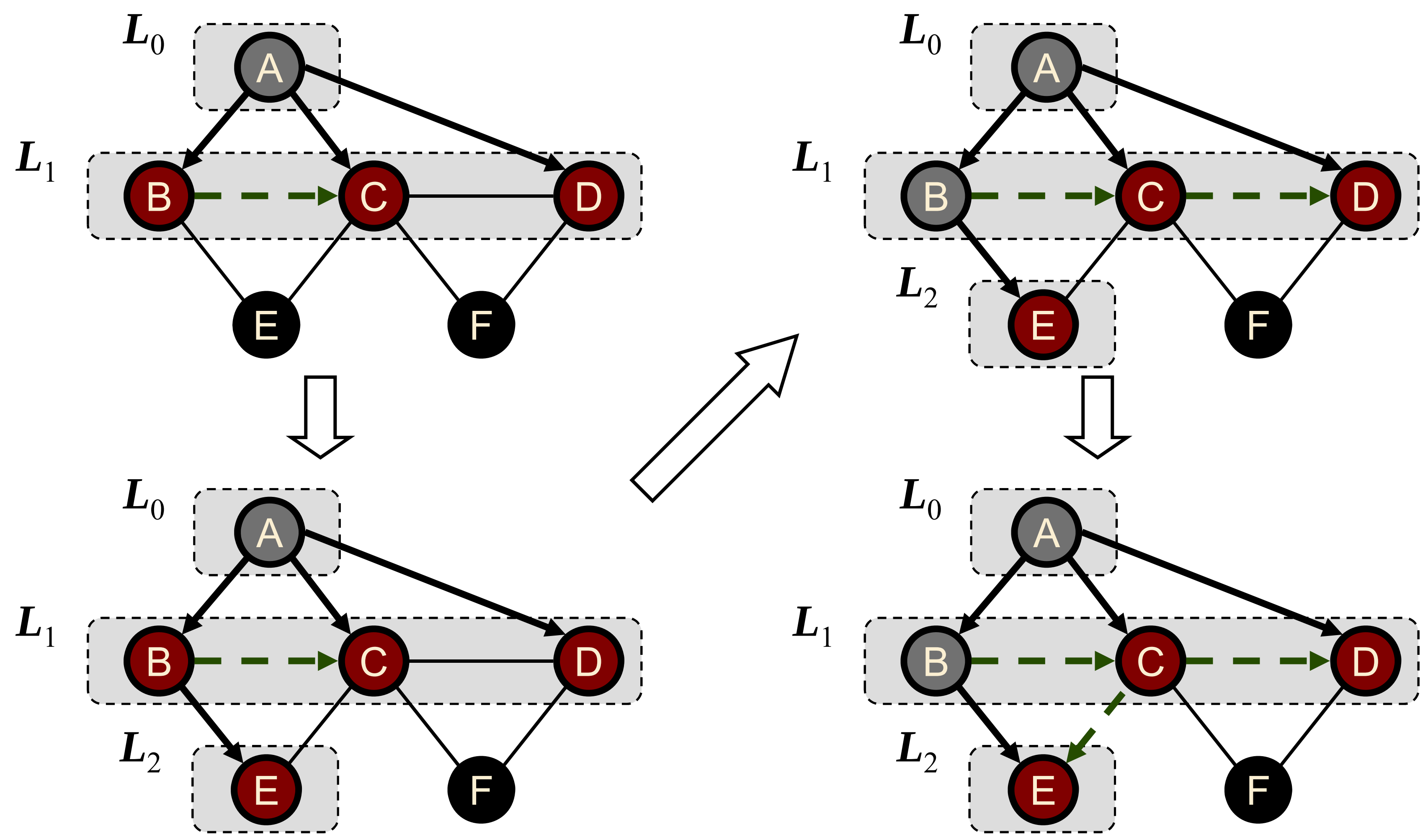
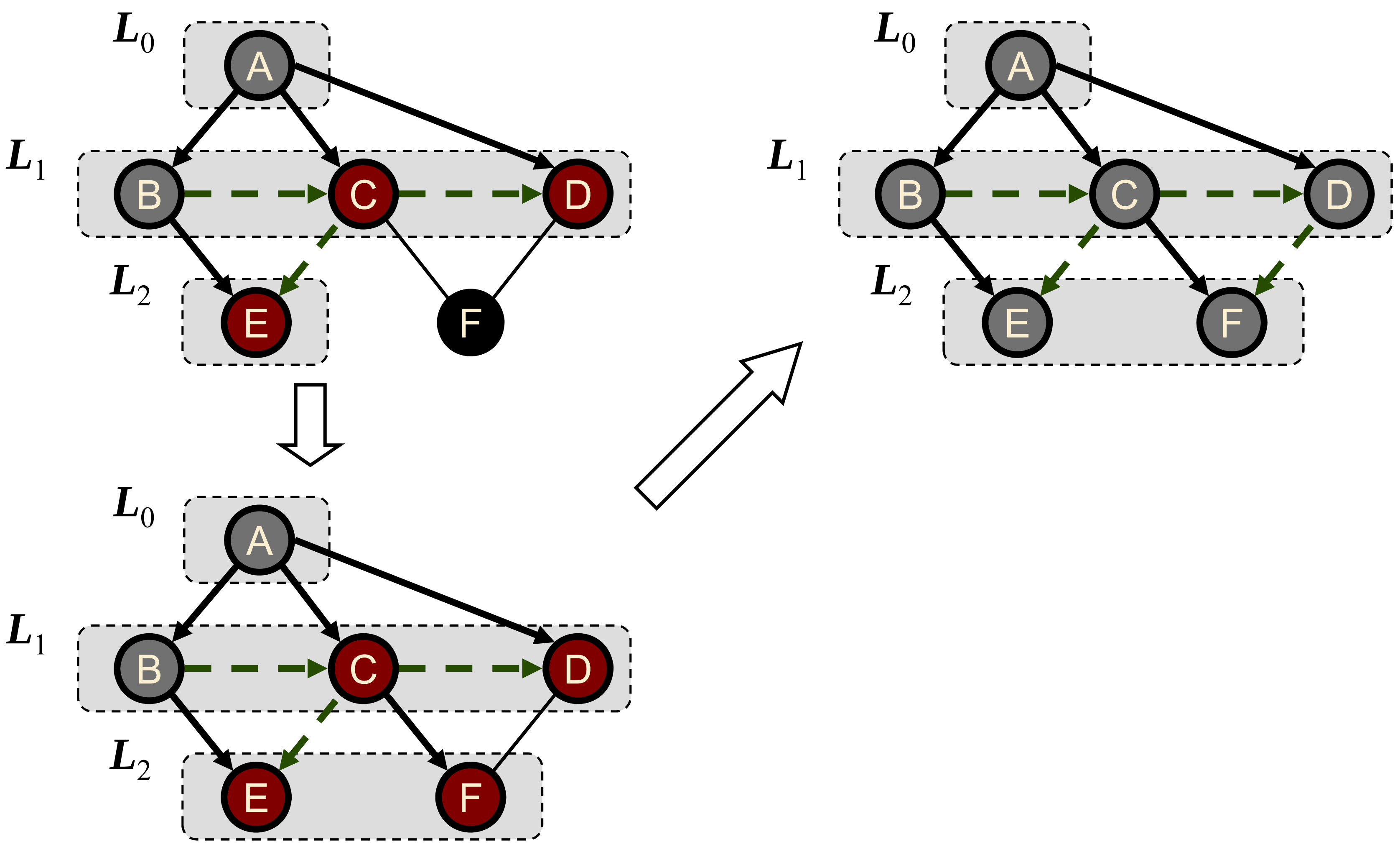
BFS Example (cont.)



BFS Example (cont.)



Properties

Notation

G_s: connected component of s

Property 1

$BFS(G, s)$ visits all the vertices and edges of G_s

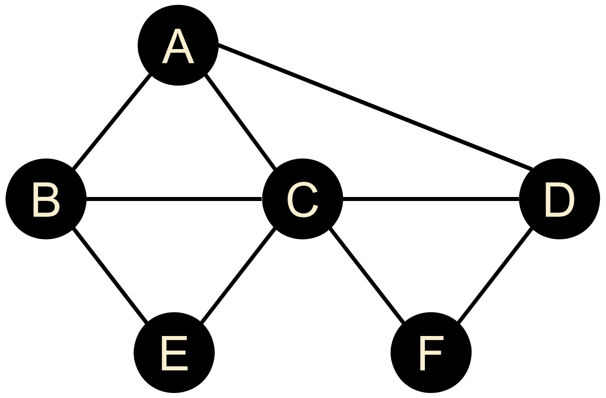
Property 2

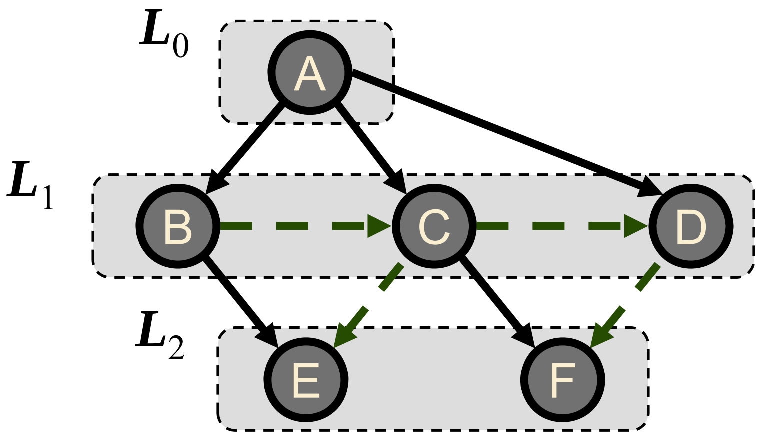
The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s

Property 3

For each vertex v in L_i

- The path of T_s from s to v has i edges
- Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled three times
  - once as BLACK (undiscovered)
  - once as RED (discovered, on queue)
  - once as GRAY (finished)
- Each edge is considered twice (for an undirected graph)
- Each vertex is placed on the queue once
- Thus BFS runs in $O(|V|+|E|)$ time provided the graph is represented by an adjacency list structure

Applications

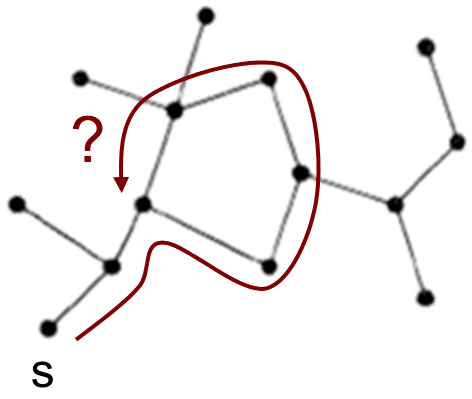
- BFS traversal can be specialized to solve the following problems in $O(|V|+|E|)$ time:
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

Outline

- BFS Algorithm
- **BFS Application: Shortest Path on an unweighted graph**
- Unweighted Shortest Path: Proof of Correctness

Application: Shortest Paths on an Unweighted Graph

- **Goal:** To recover the shortest paths from a source node *s* to all other reachable nodes *v* in a graph.
  - The length of each path and the paths themselves are returned.
- **Notes:**
  - There are an exponential number of possible paths
  - Analogous to level order traversal for trees
  - This problem is harder for general graphs than trees because of cycles!



Breadth-First Search

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ shortest path distance $\delta(s, v)$ from s to v, $\forall v \in V$.

$\pi[v] = u$ such that (u, v) is last edge on a shortest path from s to v.

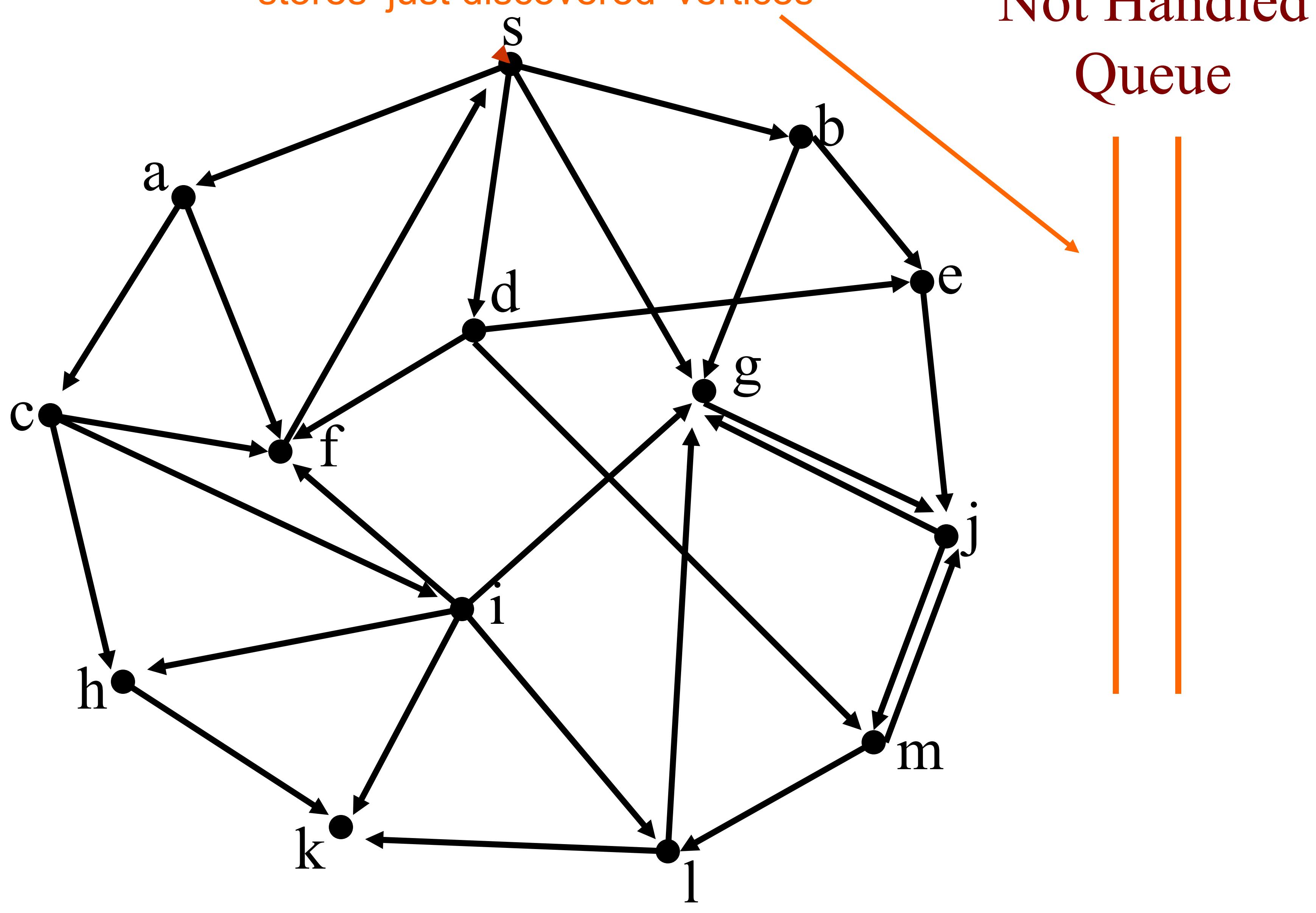
- Idea: send out search 'wave' from s.
- Keep track of progress by colouring vertices:
    - **Undiscovered** vertices are coloured **black**
    - **Just discovered** vertices (on the wavefront) are coloured **red.**
    - **Previously discovered** vertices (behind wavefront) are coloured **grey.**

BFS Algorithm with Distances and Predecessors

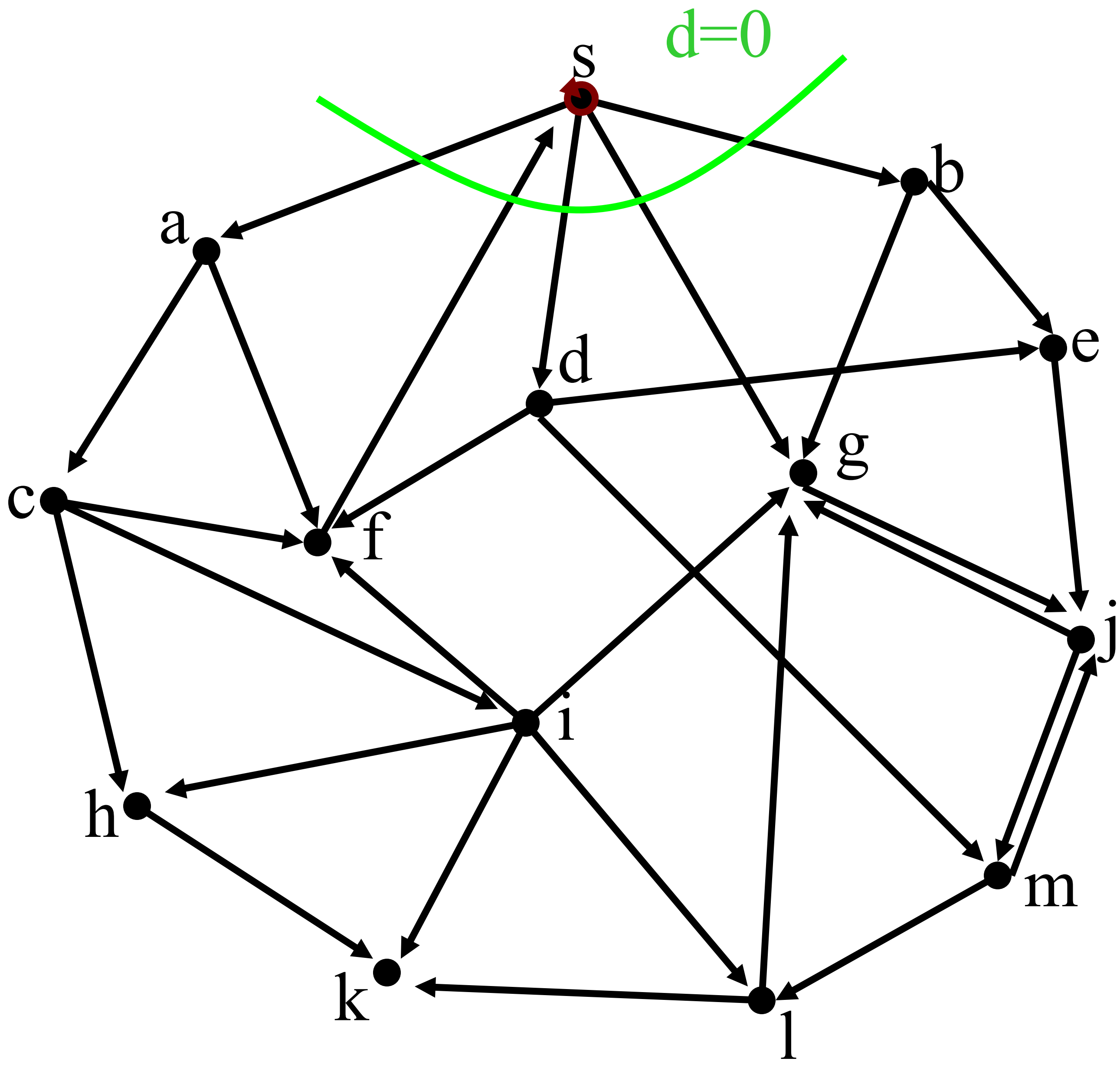
```
BFS(G,s)
Precondition: $G$ is a graph, $s$ is a vertex in $G$
Postcondition: $d[u] =$ shortest distance $\delta[u]$ and
$\pi$[u] = predecessor of u on shortest path from $s$ to each vertex $u$ in $G$
    for each vertex u $\in V[G]$
        $d[u] \leftarrow \infty$
        $\pi[u] \leftarrow$ null
        color[u] = BLACK //initialize vertex
    colour[s] $\leftarrow$ RED
    $d[s] \leftarrow 0$
    Q.enqueue($s$)
    while Q $\neq \emptyset$
        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
            if color[$v$] = BLACK
                colour[v] $\leftarrow$ RED
                $d[v] \leftarrow d[u] + 1$
                $\pi[v] \leftarrow u$
                Q.enqueue($v$)
        $colour[u] \leftarrow GRAY$
```

BFS

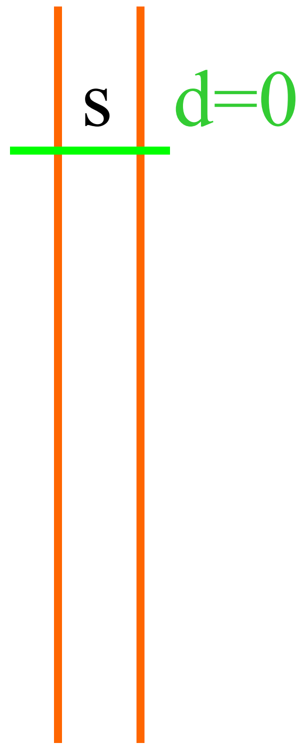
First-In First-Out (FIFO) queue
stores 'just discovered' vertices



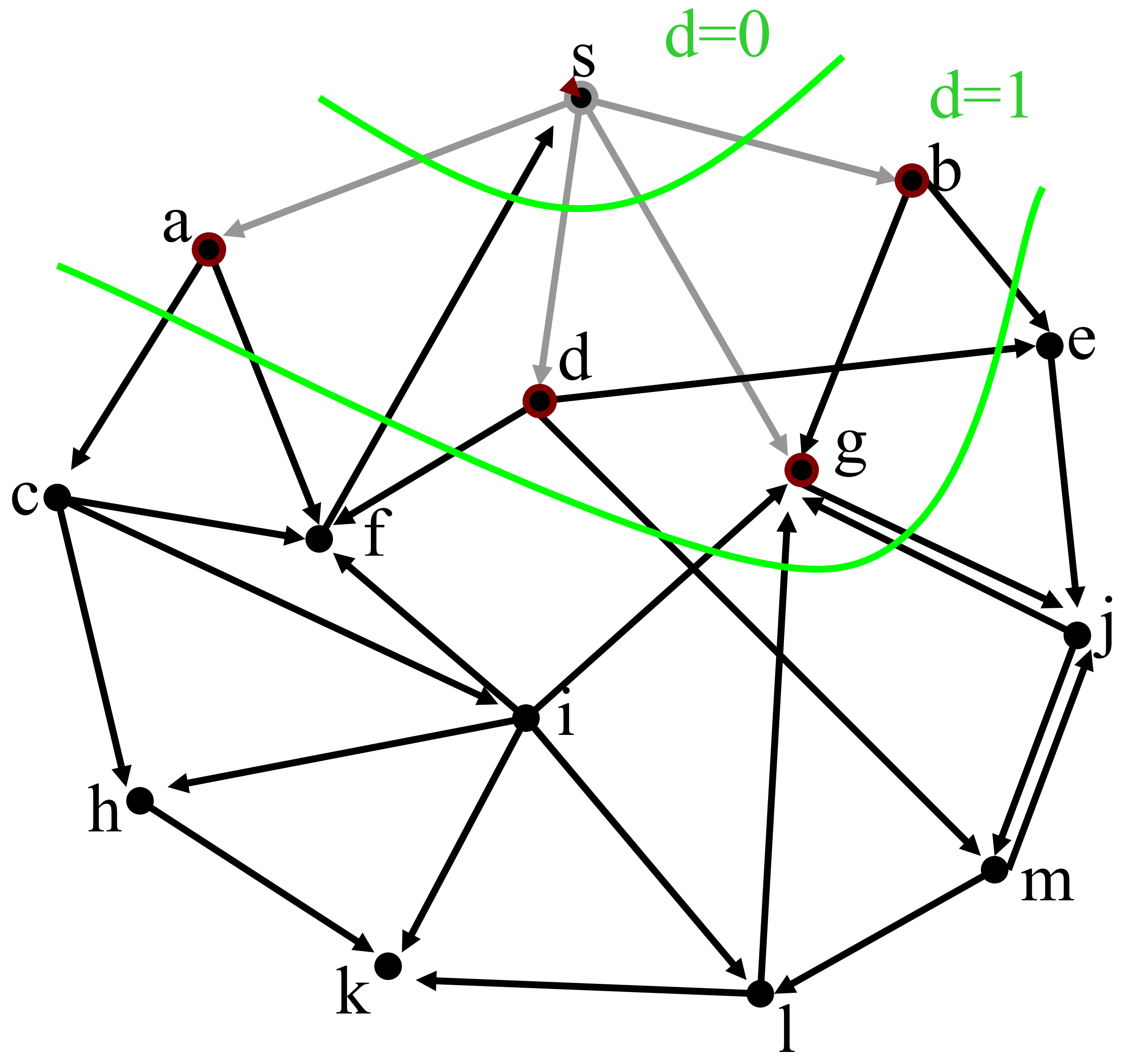
BFS



Found
Not Handled
Queue



BFS



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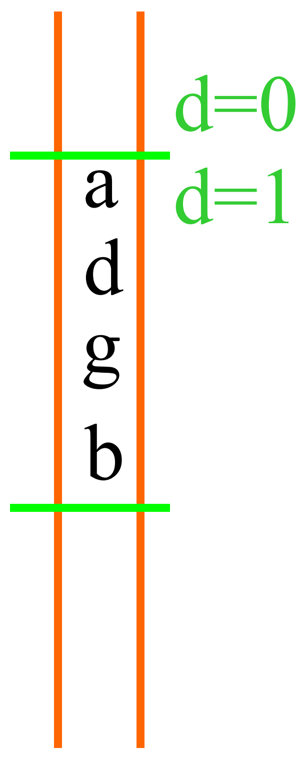
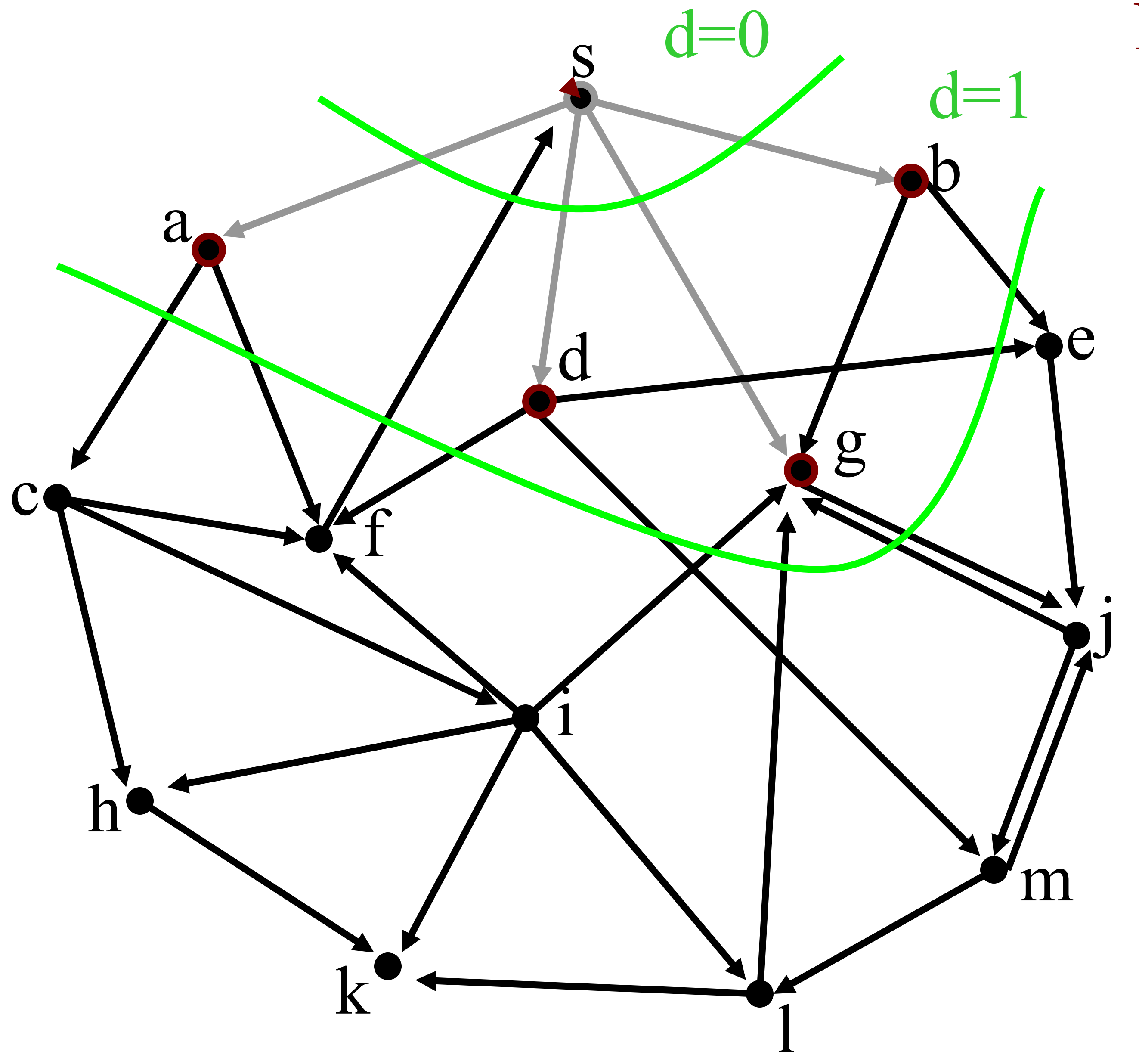


BFS



Found
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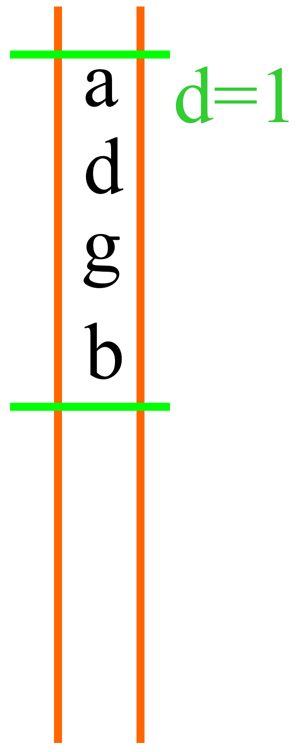
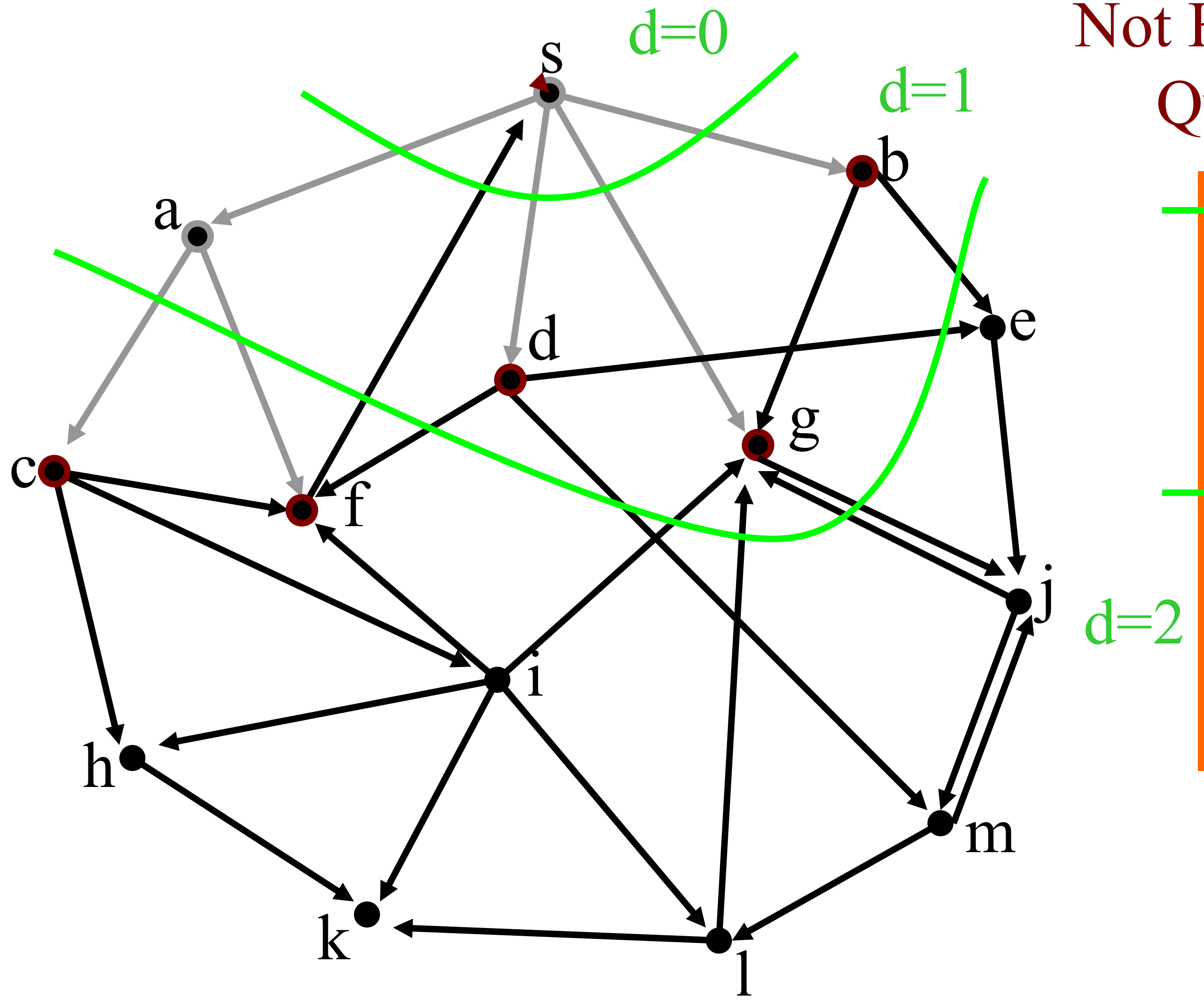


BFS



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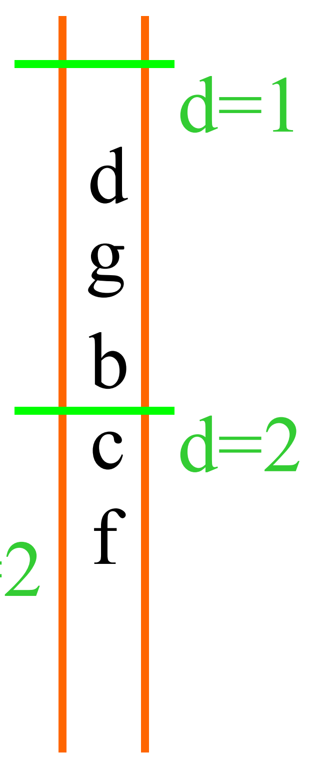

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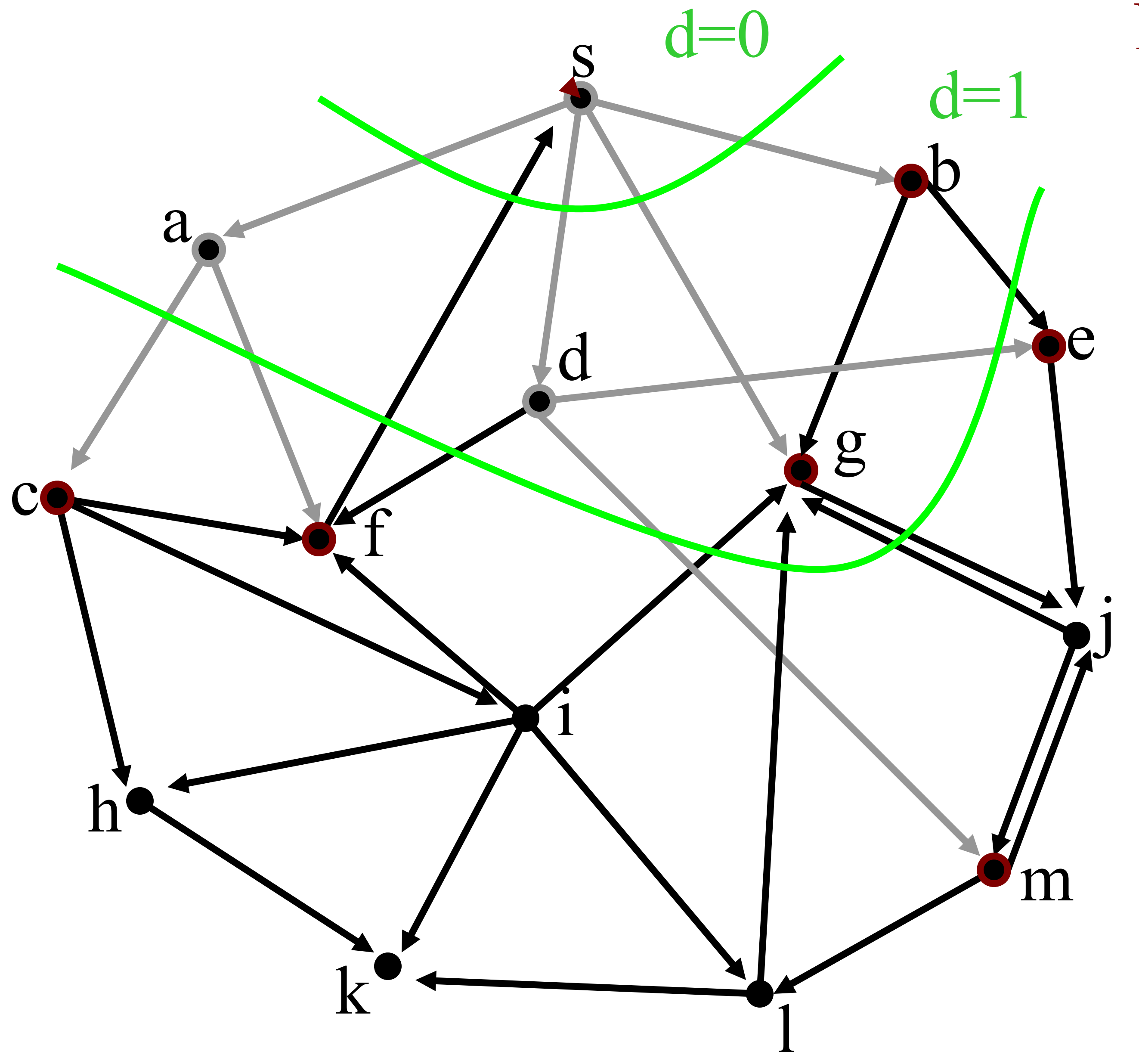

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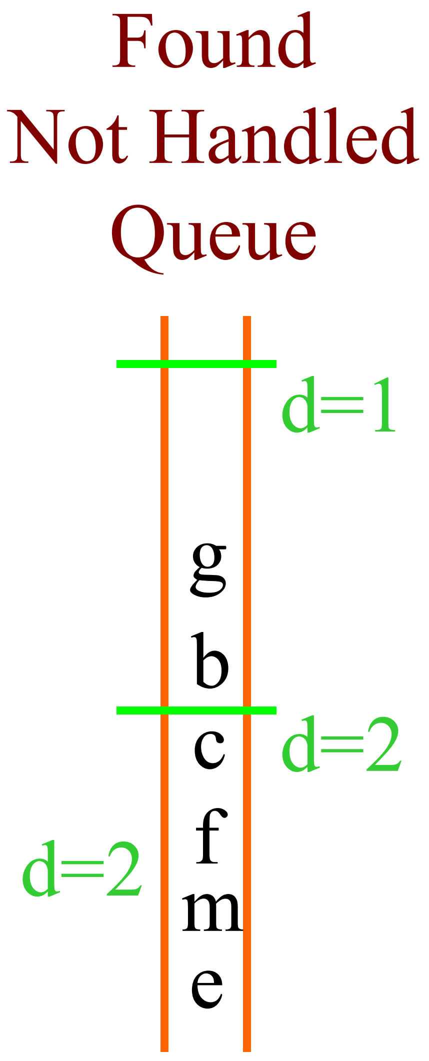

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d=1
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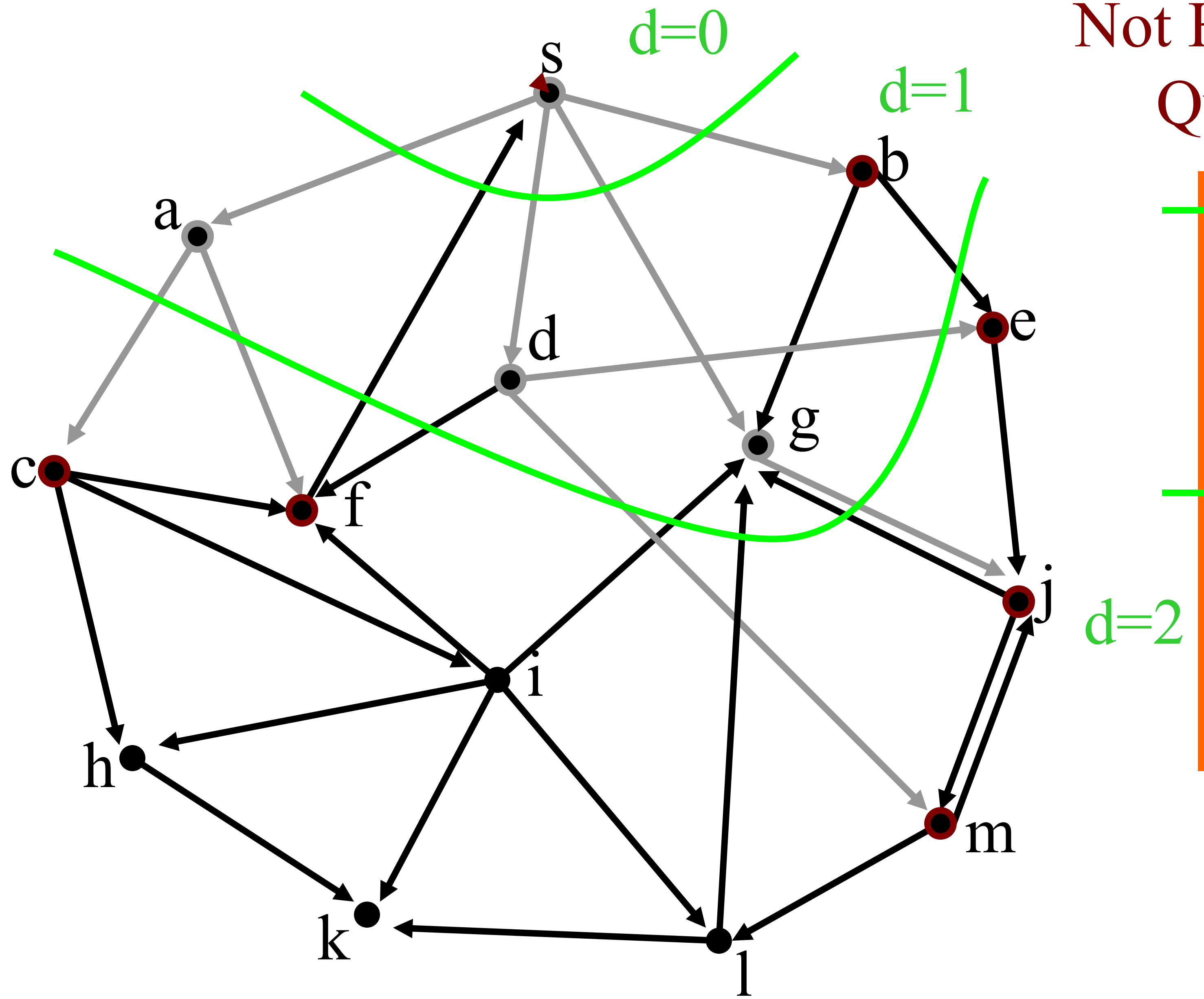
BFS



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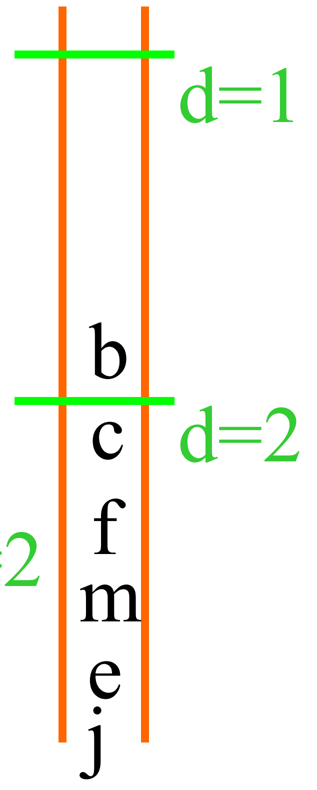

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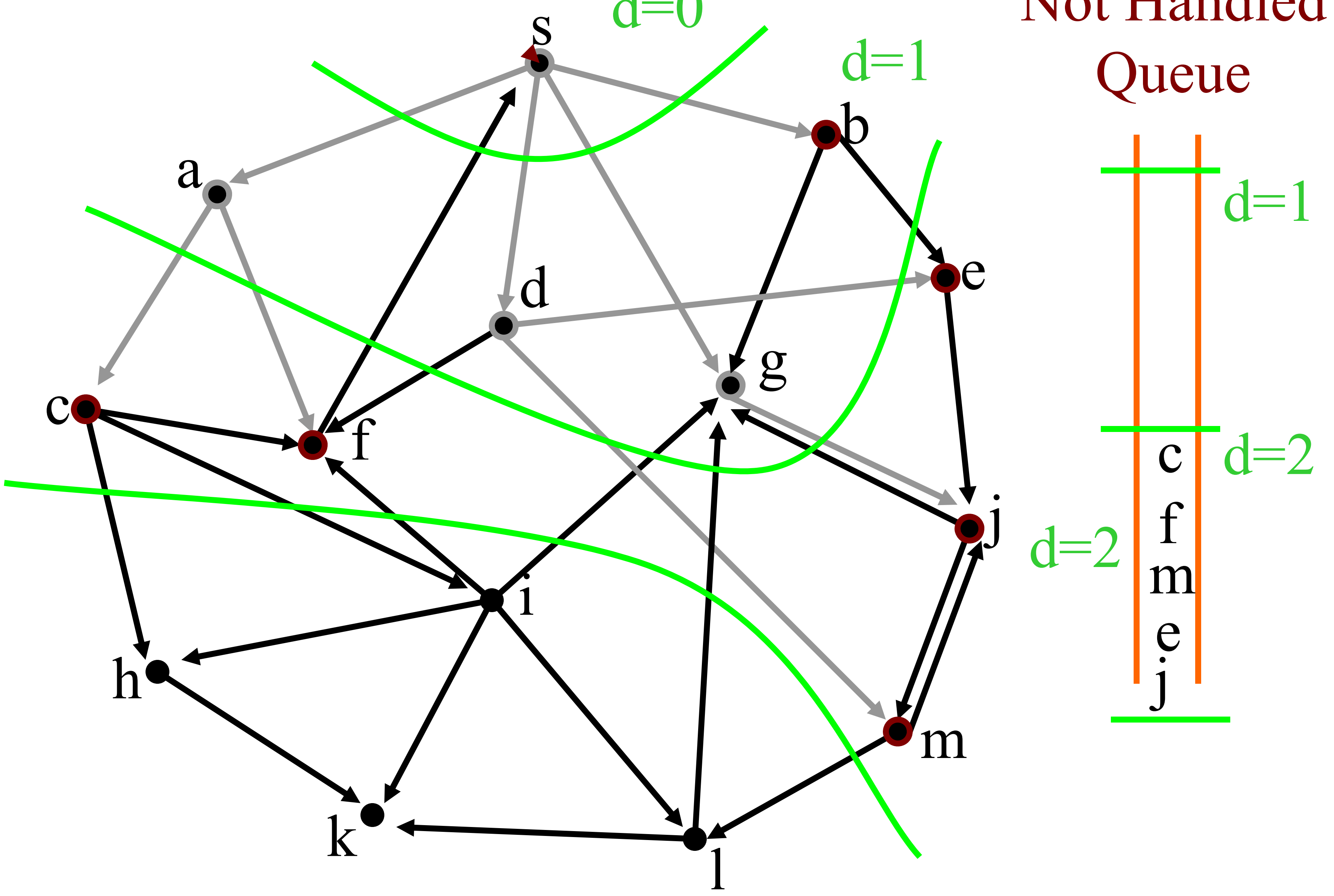

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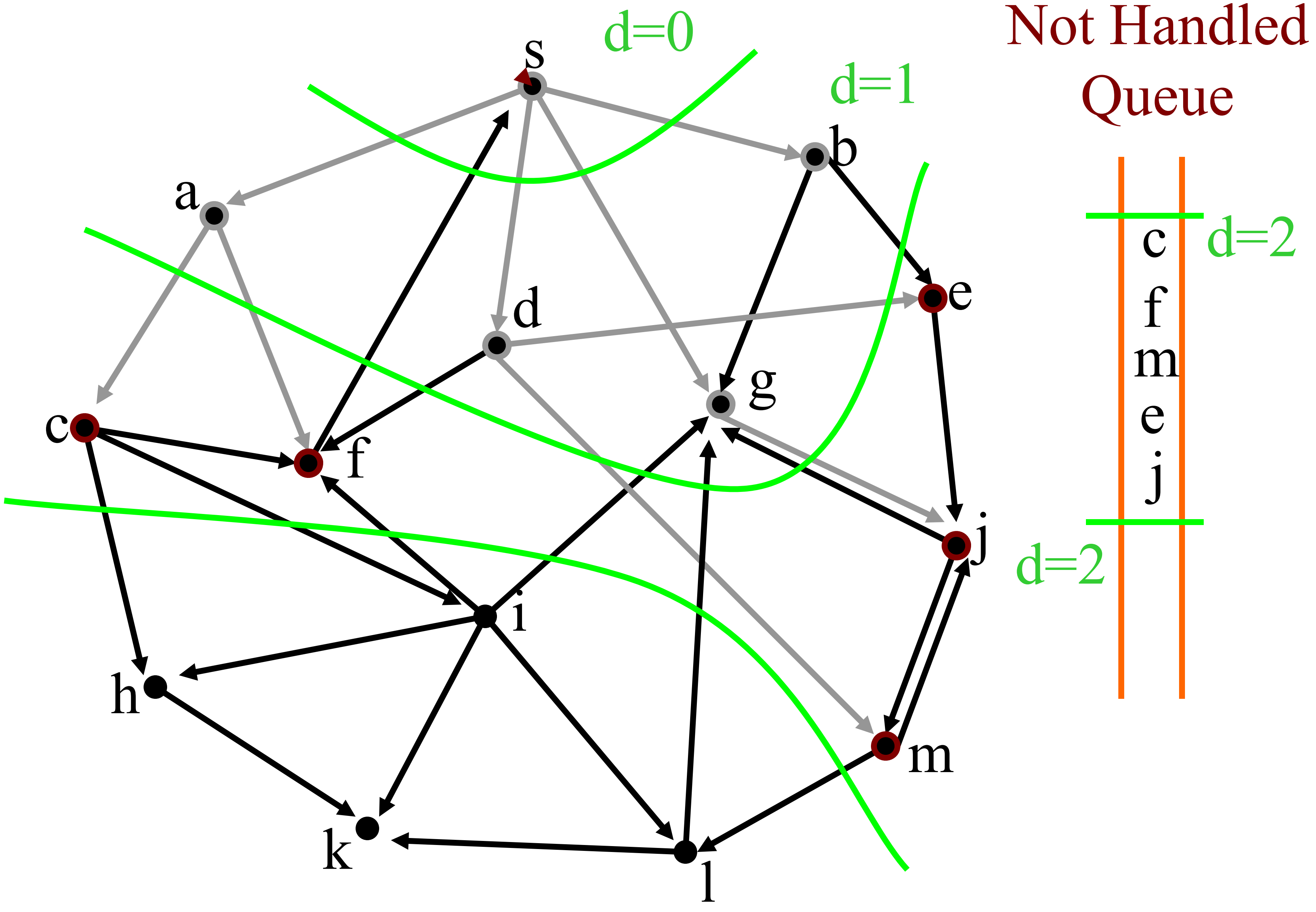

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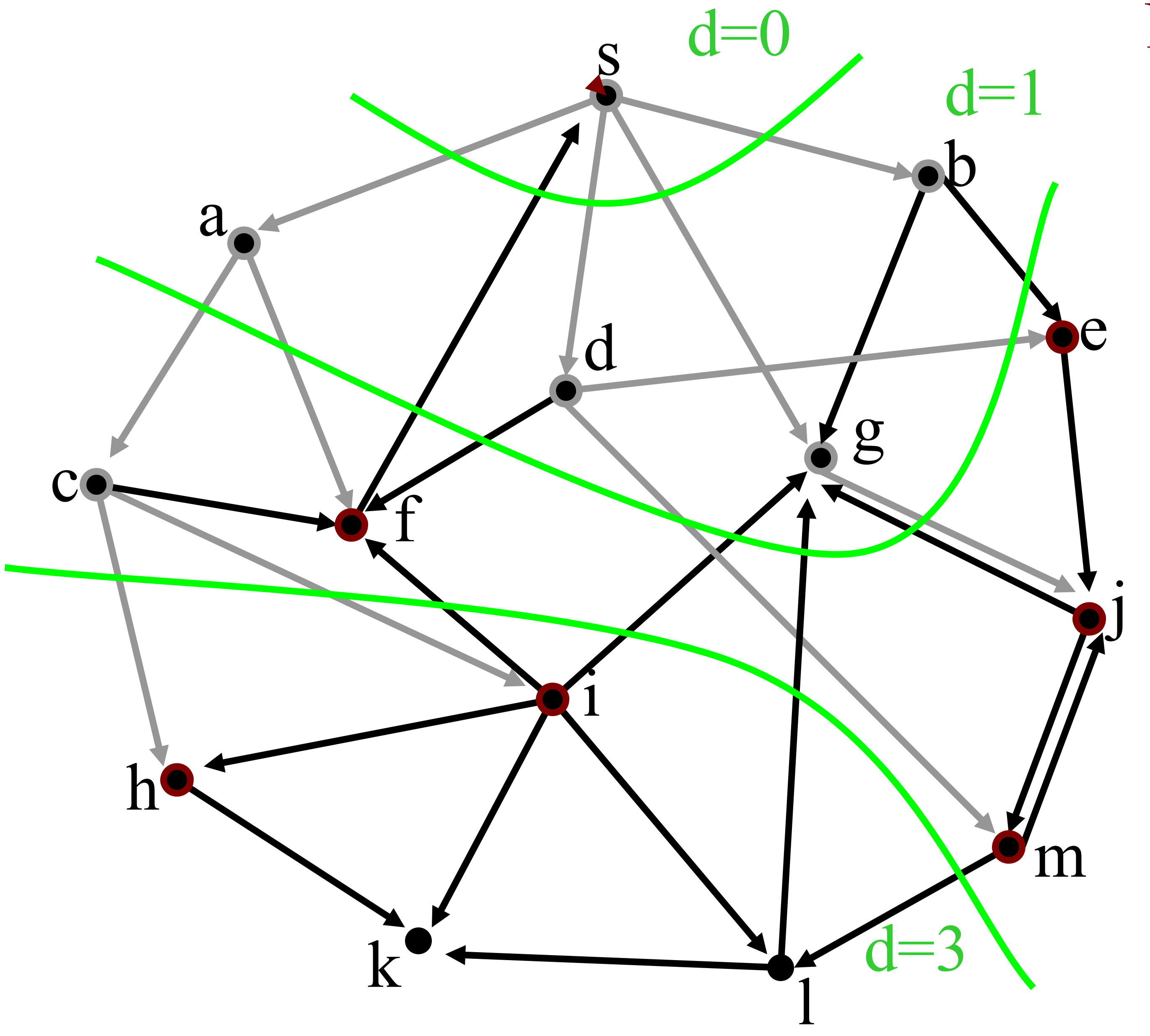

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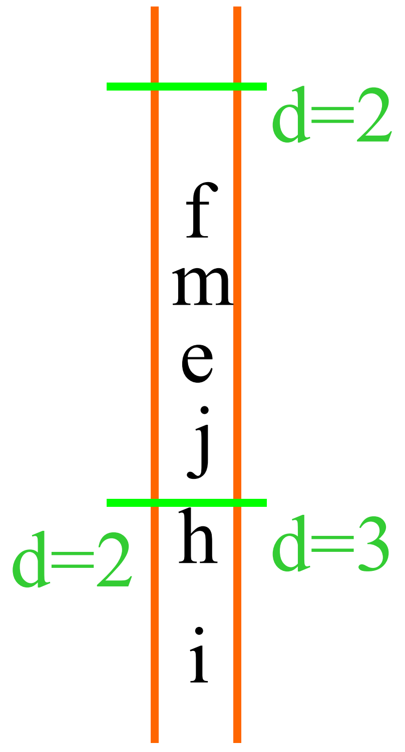

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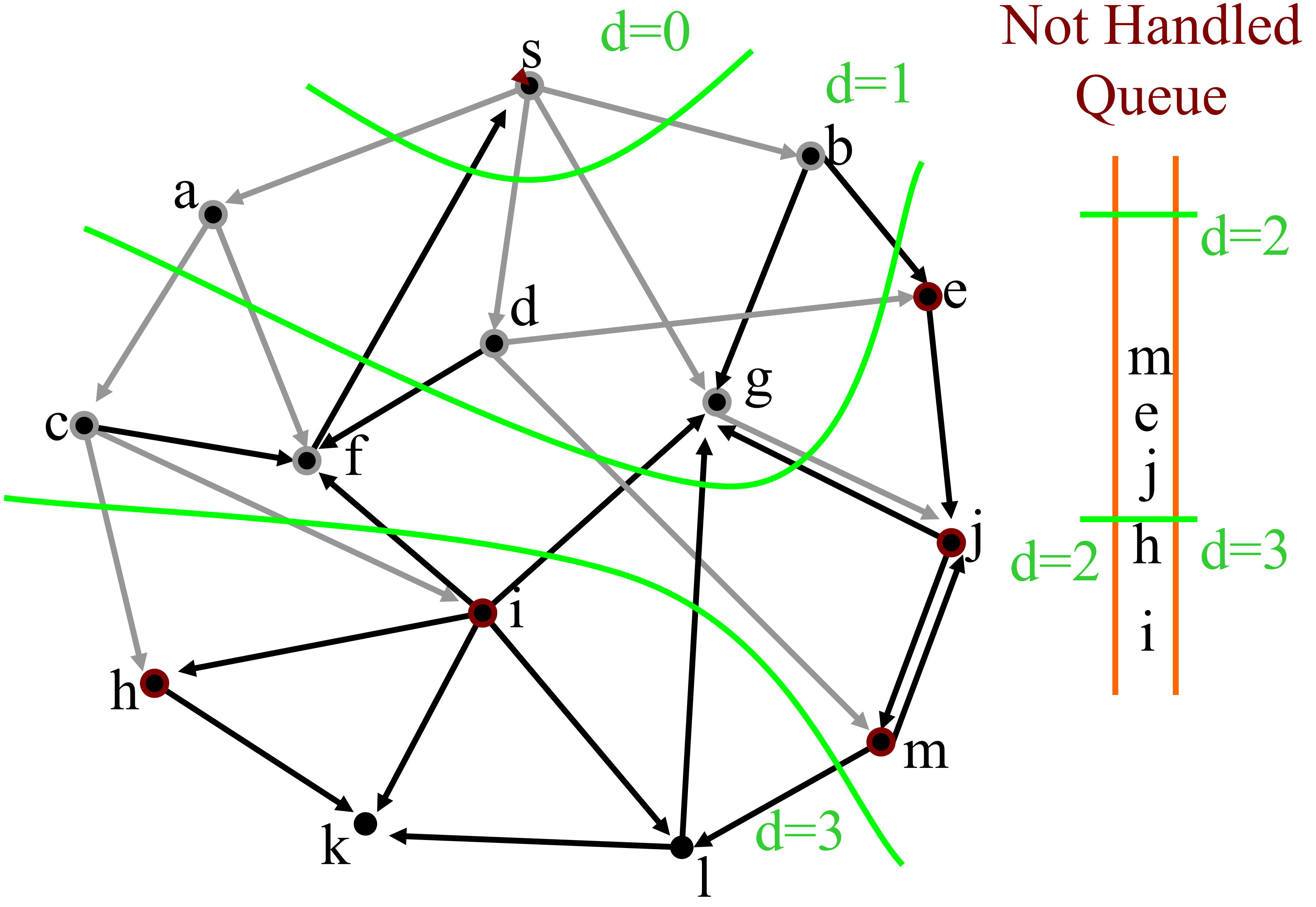
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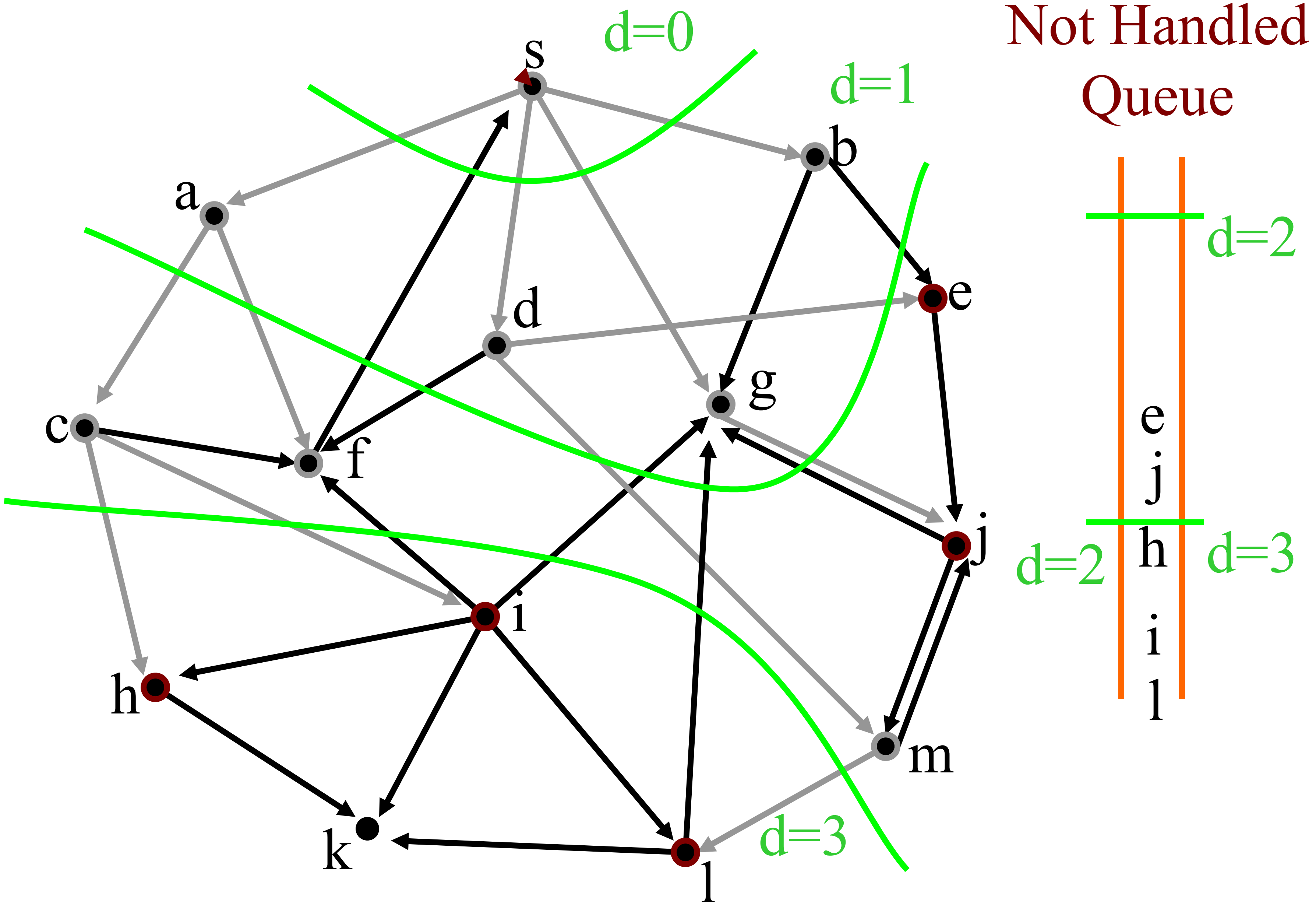
BFS



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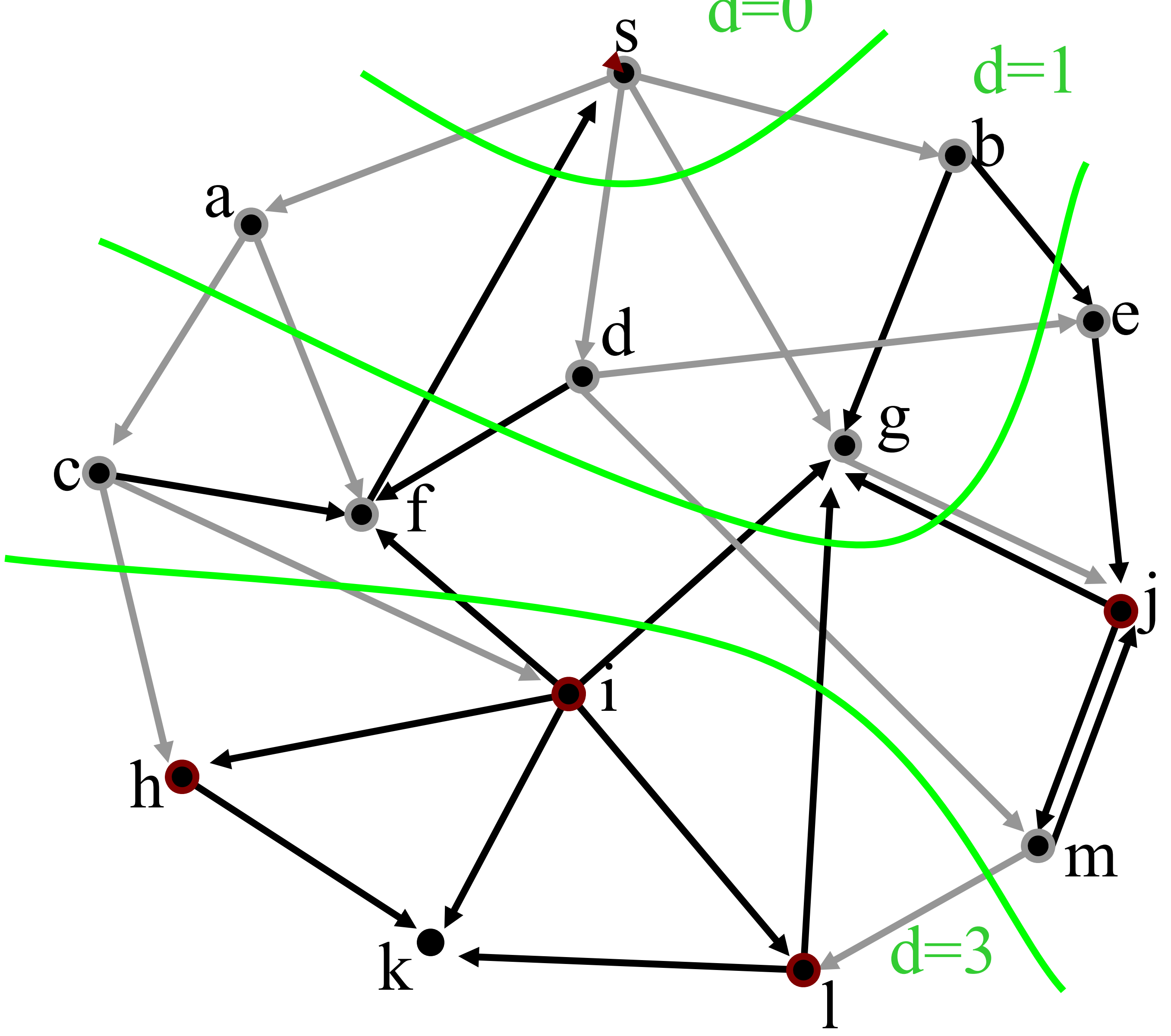
BFS



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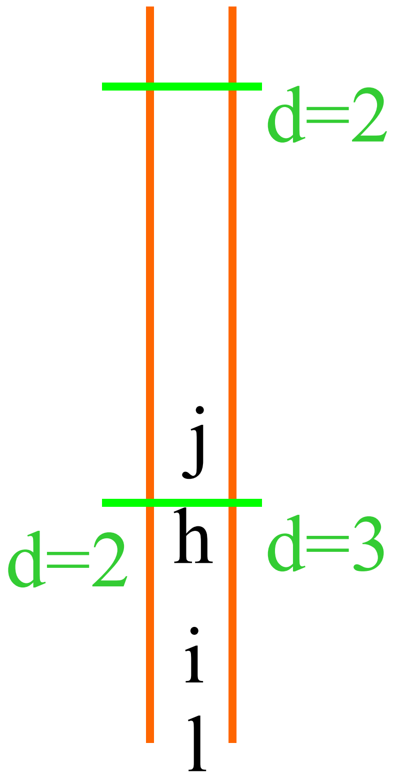

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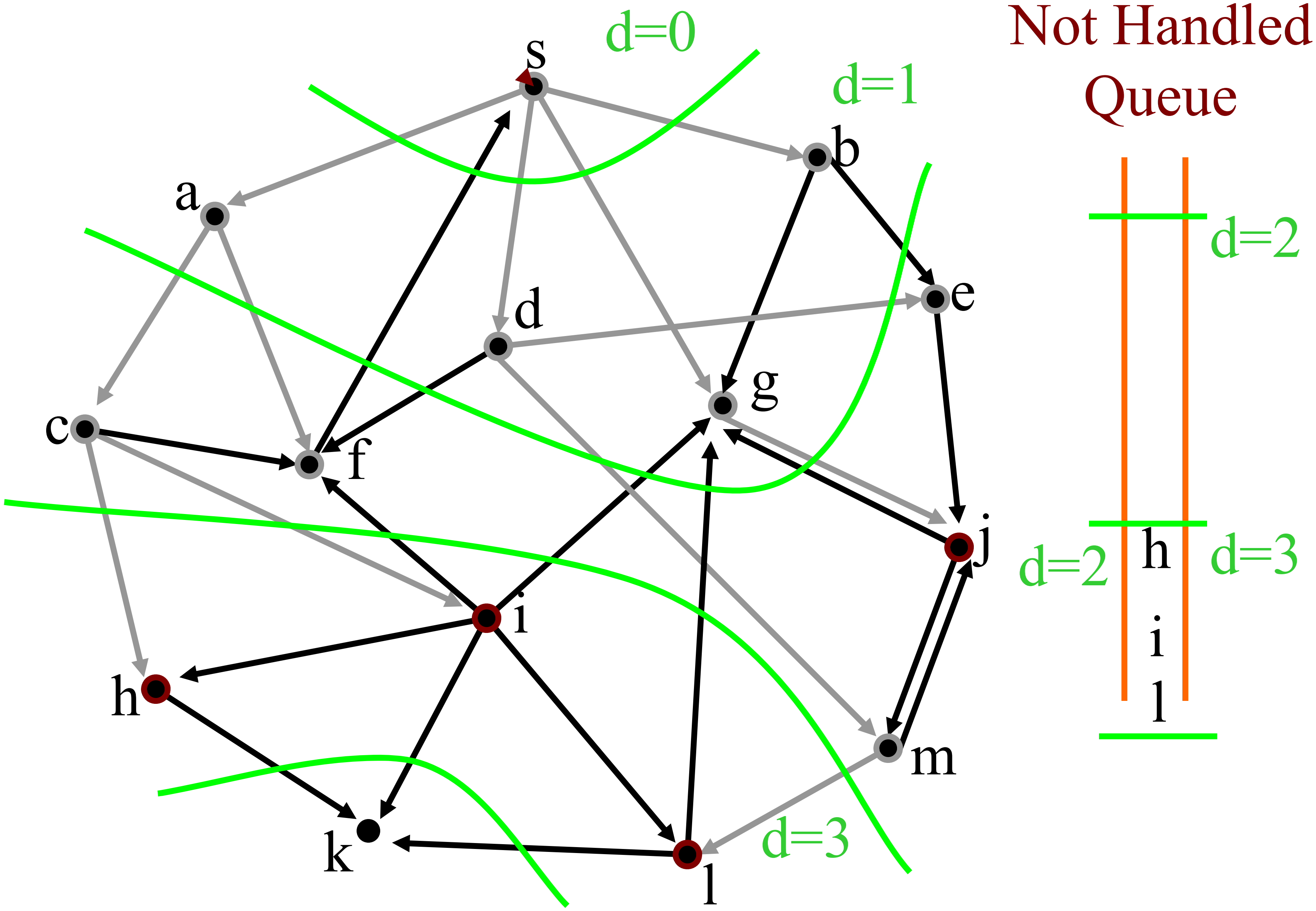
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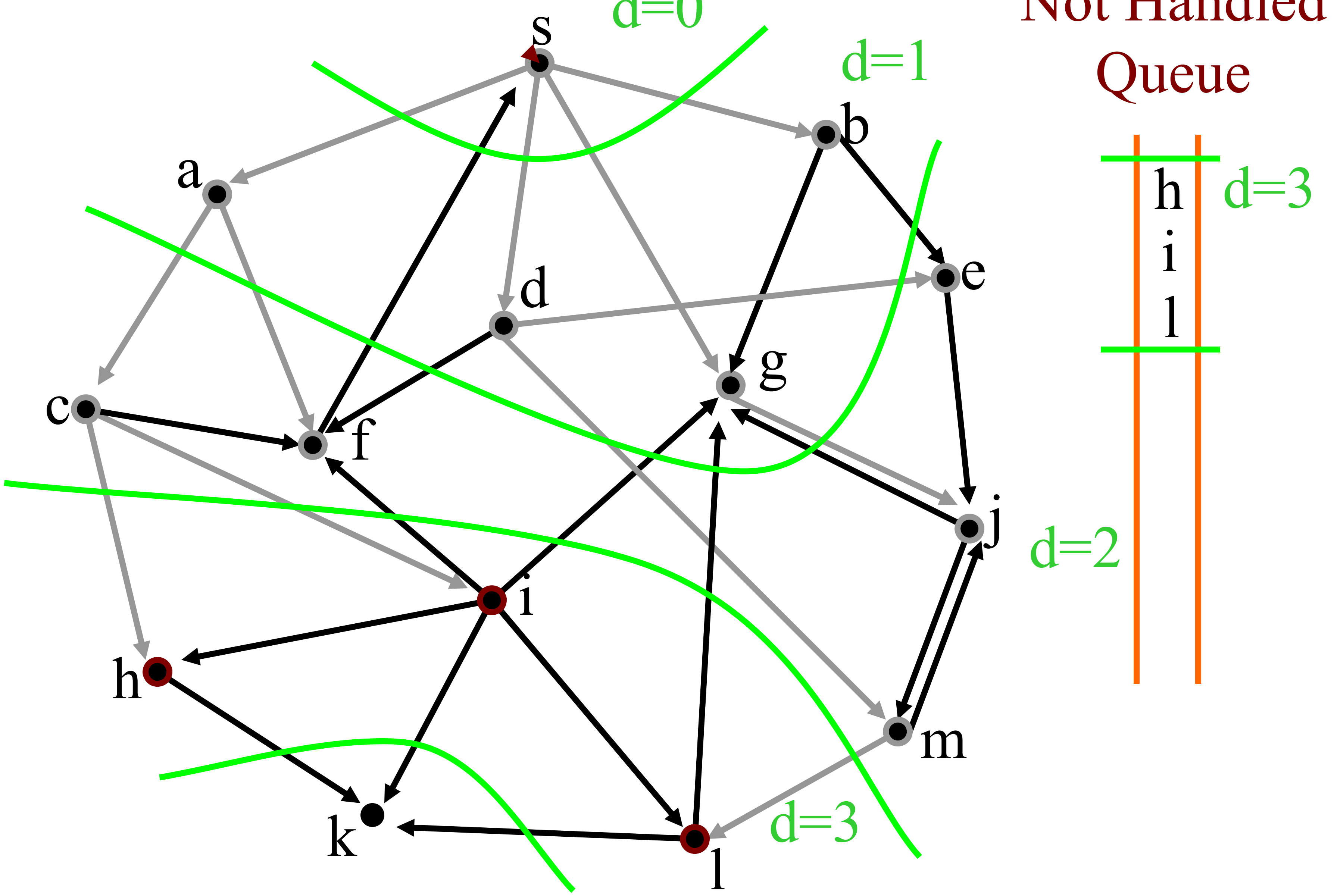
BFS



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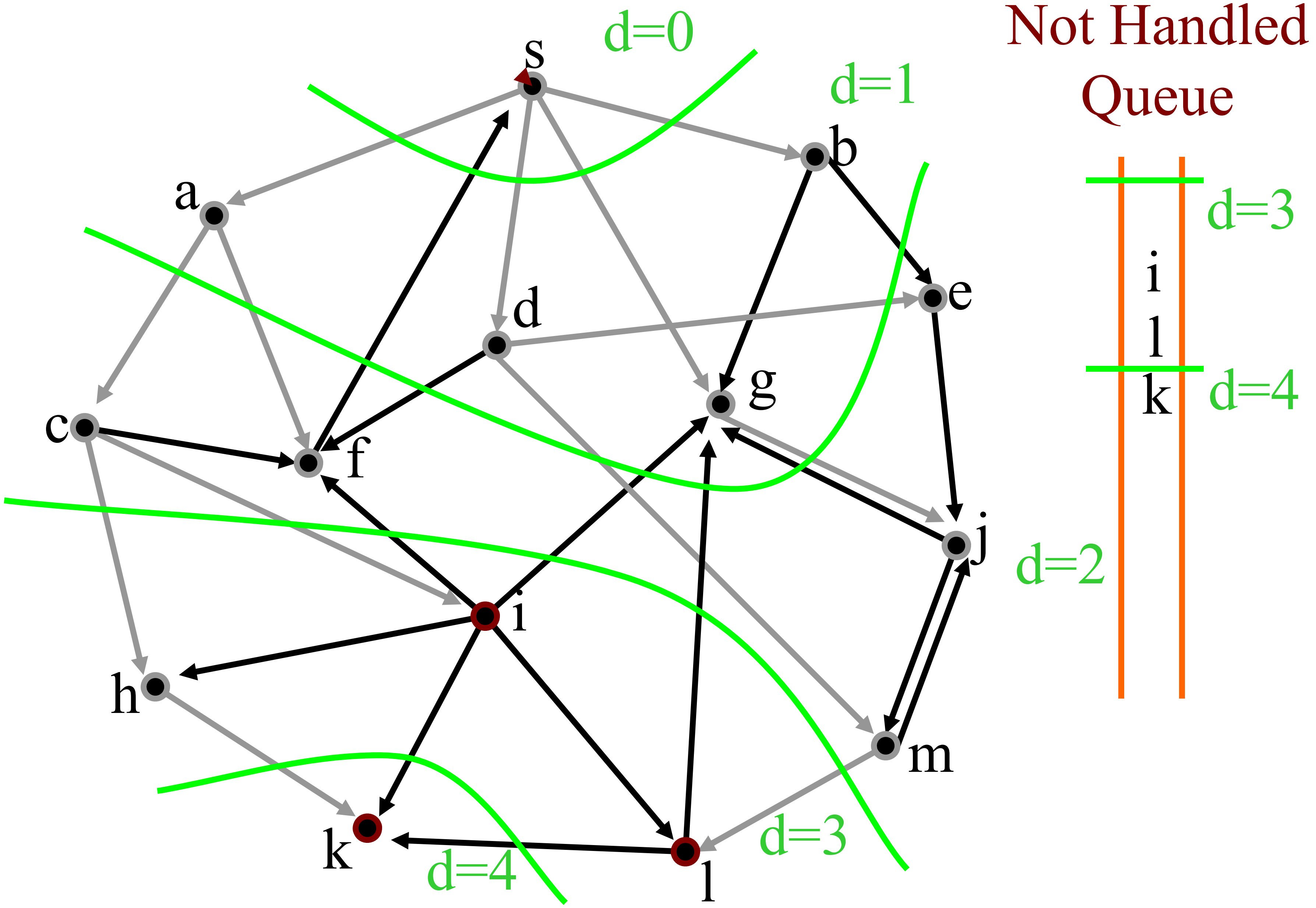
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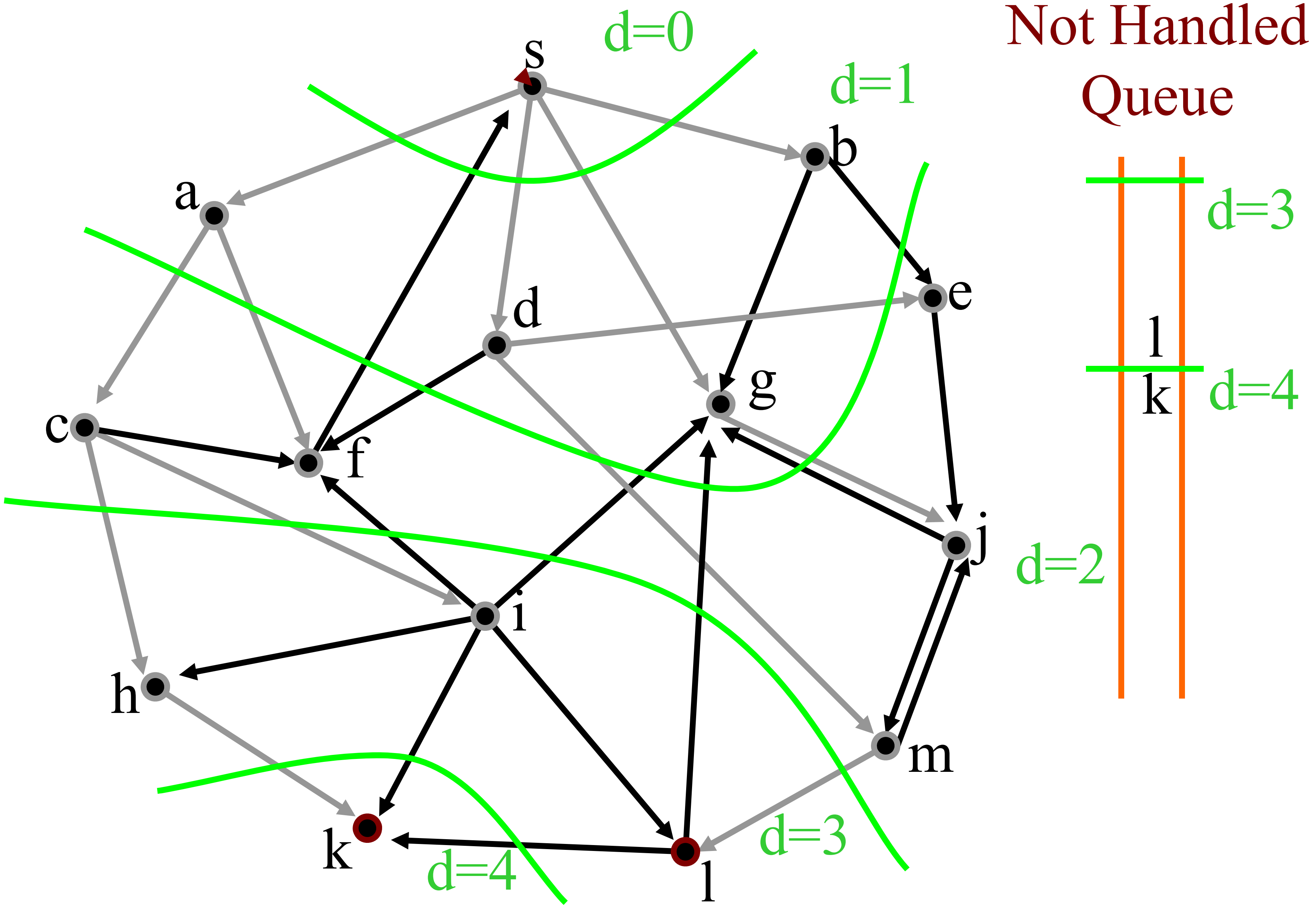
BFS



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Queue
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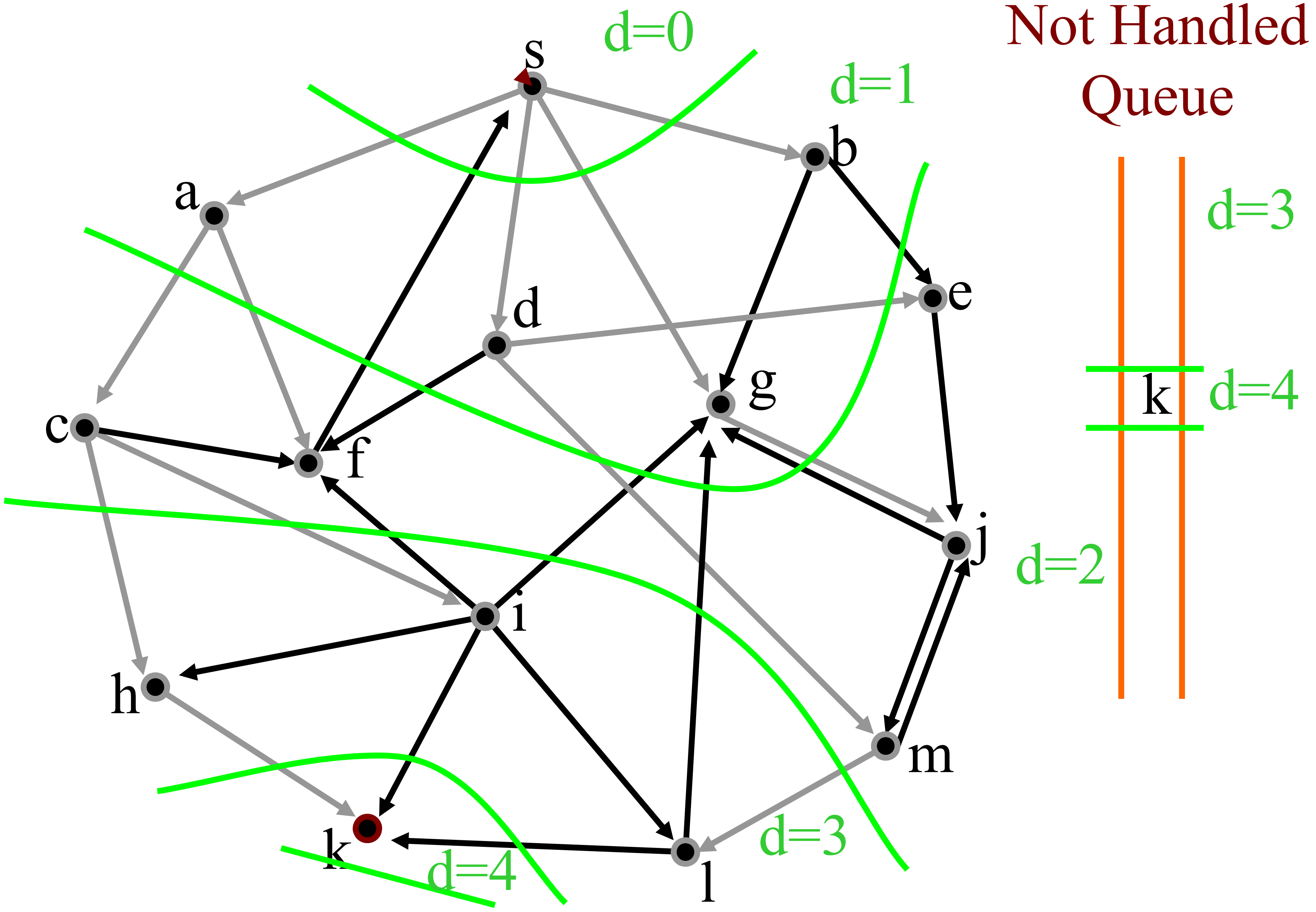
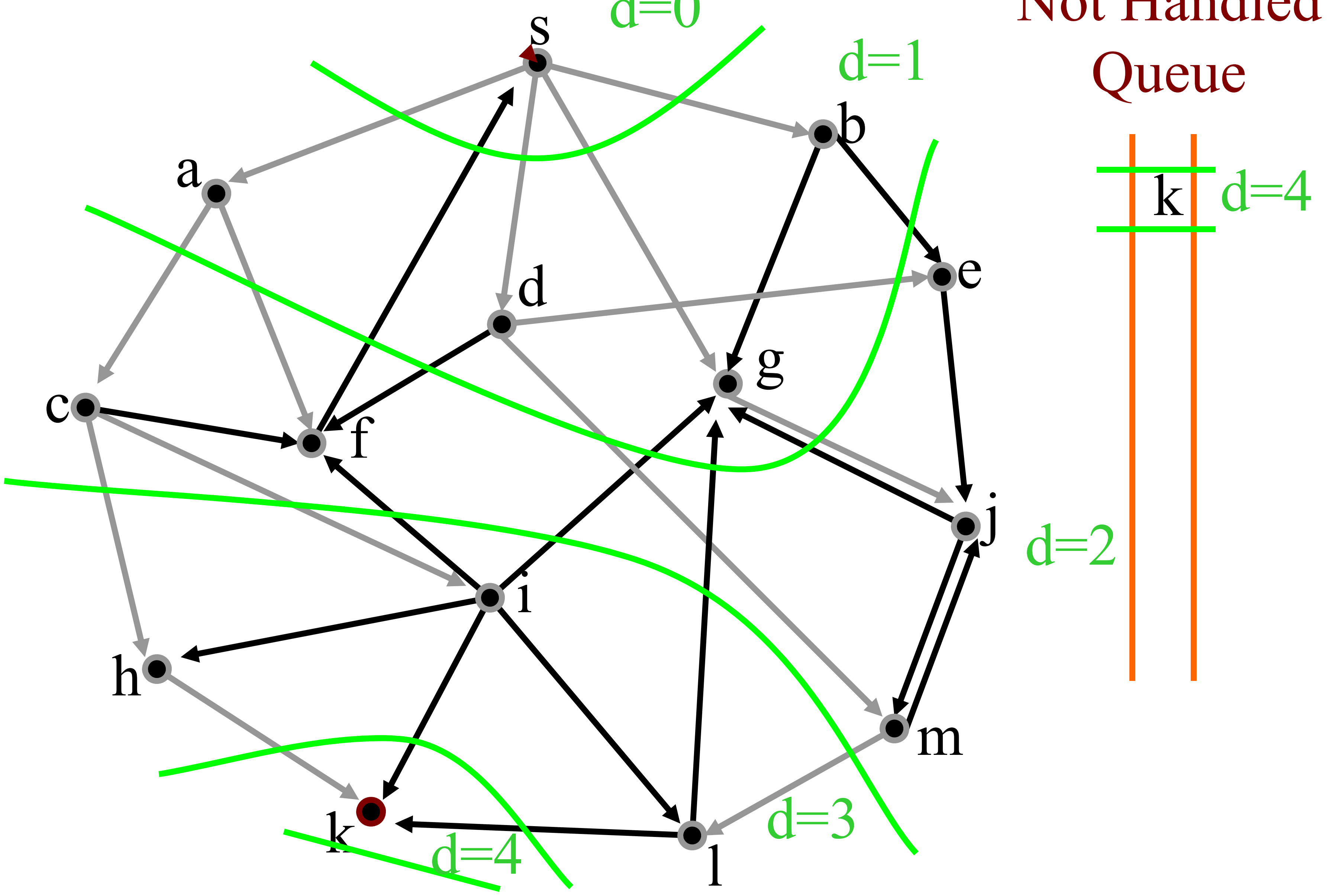
BFS



BFS



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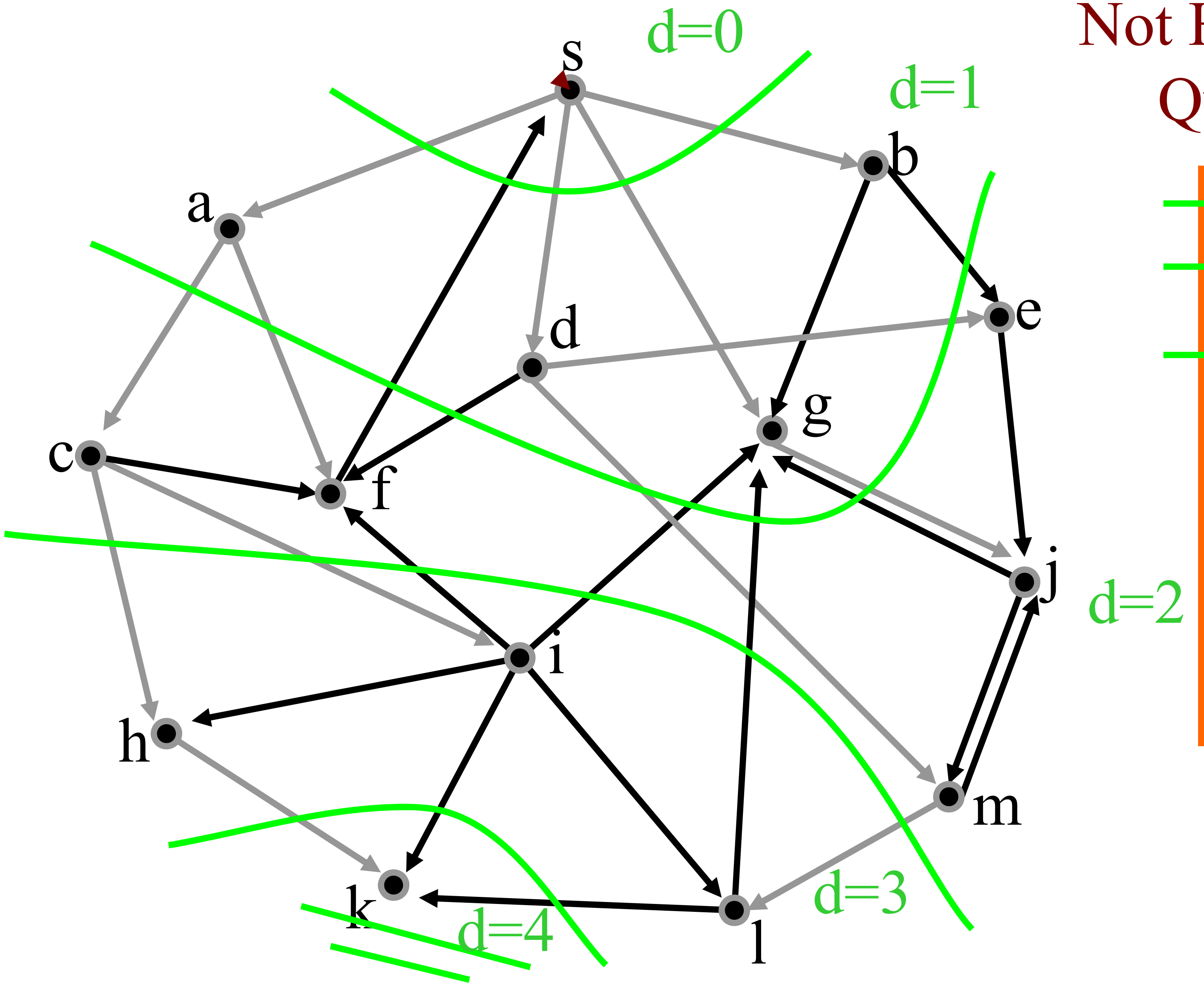

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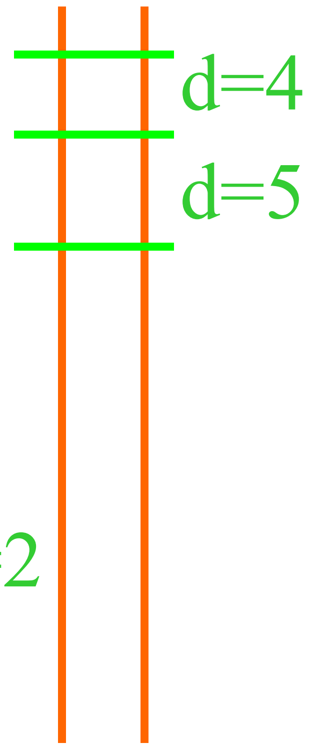


Breadth-First Search Algorithm: Properties

```
BFS(G,s)
Precondition: $G$ is a graph, $s$ is a vertex in $G$
Postcondition: $d[u] =$ shortest distance $\delta[u]$ and
$\pi[u]$ = predecessor of u on shortest paths from $s$ to each vertex $u$ in $G$
    for each vertex u $\in V[G]$
        $d[u] \leftarrow \infty$
        $\pi[u] \leftarrow$ null
        color[u] = BLACK //initialize vertex
    colour[s] $\leftarrow$ RED
    $d[s] \leftarrow 0$
    Q.enqueue($s$)
    while Q $\neq \emptyset$
        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
            if color[$v$] = BLACK
                colour[v] $\leftarrow$ RED
                $d[v] \leftarrow d[u] + 1$
                $\pi[v] \leftarrow u$
                Q.enqueue($v$)
        $colour[u] \leftarrow GRAY$
```

- Q is a FIFO queue.
- Each vertex assigned finite *d* value at most once.
- *Q* contains vertices with d values $\{i, \ldots, i, i+1, \ldots, i+1\}$
- *d* values assigned are monotonically increasing over time.

Breadth-First-Search is Greedy

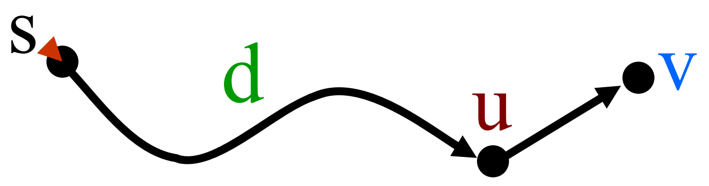
- Vertices are handled (and finished):
  - in order of their discovery (FIFO queue)
  - Smallest *d* values first

Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- **Unweighted Shortest Path: Proof of Correctness**

Correctness

Basic Steps:



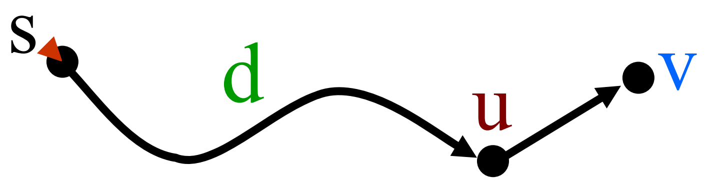
The shortest path to u has length d

& there is an edge from u to v

There is a path to v with length d+1.

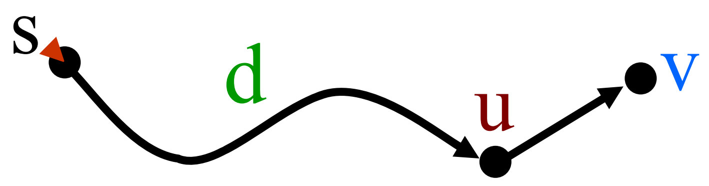
Correctness: Basic Intuition

- When we discover *v*, how do we know there is not a shorter path to *v*?
  - Because if there was, we would already have discovered it!



Correctness: More Complete Explanation

- Vertices are discovered in order of their distance from the source vertex *s*.
- Suppose that at time t_1 we have discovered the set V_d of all vertices that are a distance of *d* from *s*.
- Each vertex in the set V_{d+1} of all vertices a distance of *d*+1 from *s* must be adjacent to a vertex in V_d
- Thus we can correctly label these vertices by visiting all vertices in the adjacency lists of vertices in V_d.



Inductive Proof of BFS

Suppose at step i that the set of nodes S_i with distance $\delta(v) \le d_i$ have been discovered and their distance values $d[v]$ have been correctly assigned.

Further suppose that the queue contains only nodes in S_i with d values of d_i.

Any node v with $\delta(v) = d_i + 1$ must be adjacent to S_i.

Any node v adjacent to S_i but not in S_i must have $\delta(v) = d_i + 1$.

At step $i+1$, all nodes on the queue with d values of d_i are dequeued and processed.

In so doing, all nodes adjacent to S_i are discovered and assigned d values of $d_i + 1$.

Thus after step $i+1$, all nodes v with distance $\delta(v) \le d_i + 1$ have been discovered and their distance values $d[v]$ have been correctly assigned.

Furthermore, the queue contains only nodes in S_i with d values of $d_i + 1$.

Correctness: Formal Proof

Input: Graph $G = (V, E)$ (directed or undirected) and source vertex $s \in V$.

Output:

$d[v] =$ distance $\delta(v)$ from s to v, $\forall v \in V$.

$\pi[v] = u$ such that (u,v) is last edge on shortest path from s to v.

Two-step proof:

On exit:

1. $d[v] \geq \delta(s,v) \forall v \in V$
2. $d[v] \not> \delta(s,v) \forall v \in V$

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

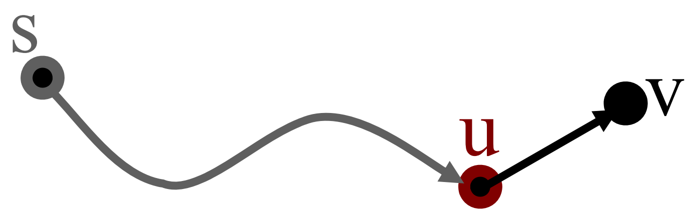
Proof: There exists a path from s to v of length $\leq d[v]$.

By Induction:

Suppose it is true for all vertices thus far discovered (red and grey).

v is discovered from some adjacent vertex u being handled.

$$\begin{aligned} \rightarrow d[v] &= d[u]+1 \\ &\geq \delta(s,u)+1 \\ &\geq \delta(s,v) \end{aligned}$$



since each vertex v is assigned a d value exactly once,

it follows that on exit, $d[v] \geq \delta(s,v) \forall v \in V$.

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Proof: There exists a path from s to v of length $\leq d[v]$.

BFS(G,s)

Precondition: G is a graph, s is a vertex in G

Postcondition: $d[u] =$ shortest distance $\delta[u]$ and

$\pi[u]$ = predecessor of u on shortest paths from s to each vertex u in G

```
for each vertex u $\in V[G]$
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        color[u] = BLACK //initialize vertex
colour[s] $\leftarrow$ RED
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        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
                if color[$v$] = BLACK
                        colour[v] $\leftarrow$ RED
                        $d[v] \leftarrow d[u] + 1$
                        $\pi[v] \leftarrow u$
                        Q.enqueue($v$)
        $colour[u] \leftarrow GRAY$
```
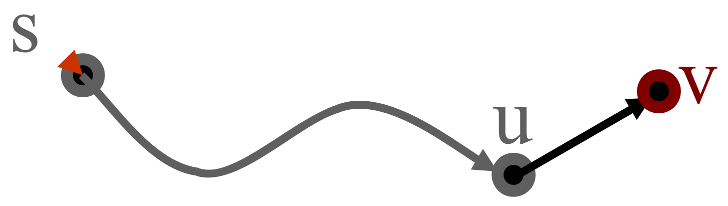

$\leftarrow$ <LI>: $d[v] \geq \delta(s,v) \forall$ 'discovered' (red or grey) $v \in V$

$\geq \delta(s,u) + 1 \geq \delta(s,v)$



Claim 2. d is never too big: $d[v] \le \delta(s,v) \forall v \in V$

Proof by contradiction:

Suppose one or more vertices receive a d value greater than δ.

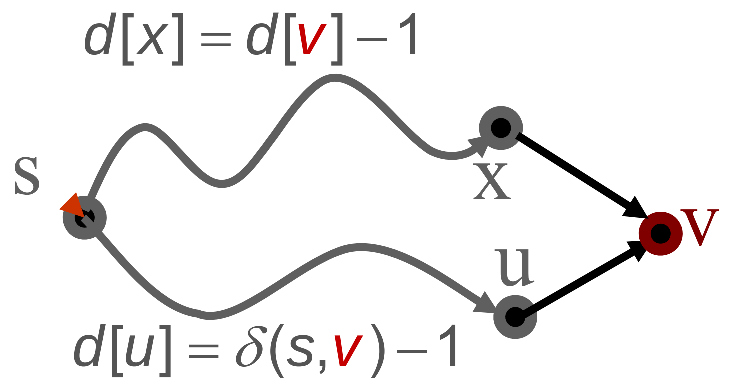
Let v be the vertex with minimum $\delta(s,v)$ that receives such a d value.

Suppose that v is discovered and assigned this d value when vertex x is dequeued.

Let u be v's predecessor on a shortest path from s to v.

Then

$$\delta(s,v) < d[v]$$
$$\to \delta(s,v) - 1 < d[v] - 1$$
$$\to d[u] < d[x]$$



Recall: vertices are dequeued in increasing order of d value.

$\to$ u was dequeued before x.

$\to d[v] = d[u] + 1 = \delta(s,v)$ Contradiction!

Correctness

Claim 1. d is never too small: $d[v] \geq \delta(s,v) \forall v \in V$

Claim 2. d is never too big: $d[v] \leq \delta(s,v) \forall v \in V$

$\Rightarrow d$ is just right: $d[v] = \delta(s,v) \forall v \in V$

Progress?

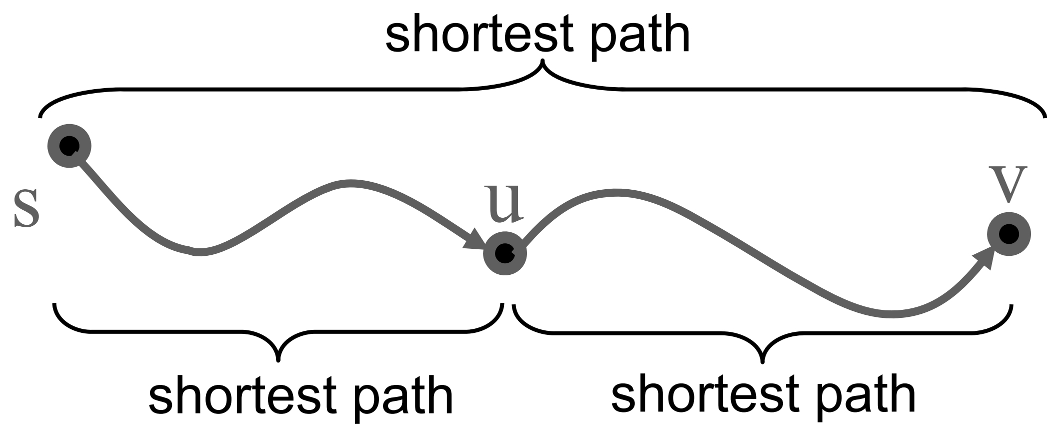
- On every iteration one vertex is processed (turns gray).

```
BFS(G,s)
Precondition: $G$ is a graph, $s$ is a vertex in $G$
Postcondition: $d[u] =$ shortest distance $\delta[u]$ and
$\pi$[u] = predecessor of u on shortest paths from $s$ to each vertex $u$ in $G$
    for each vertex u $\in V[G]$
        $d[u] \leftarrow \infty$
        $\pi[u] \leftarrow$ null
        color[u] = BLACK //initialize vertex
    colour[s] $\leftarrow$ RED
    $d[s] \leftarrow 0$
    Q.enqueue($s$)
    while Q $\neq \emptyset$
        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
            if color[$v$] = BLACK
                colour[v] $\leftarrow$ RED
                $d[v] \leftarrow d[u] + 1$
                $\pi[v] \leftarrow u$
                Q.enqueue($v$)
        $colour[u] \leftarrow GRAY$
```

Optimal Substructure Property

- The shortest path problem has the optimal substructure property:
  - Every subpath of a shortest path is a shortest path.

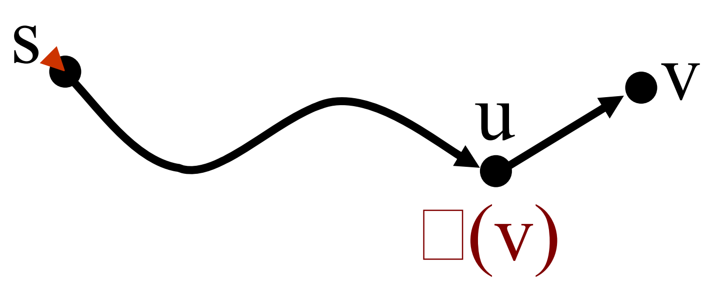
How would we prove this?

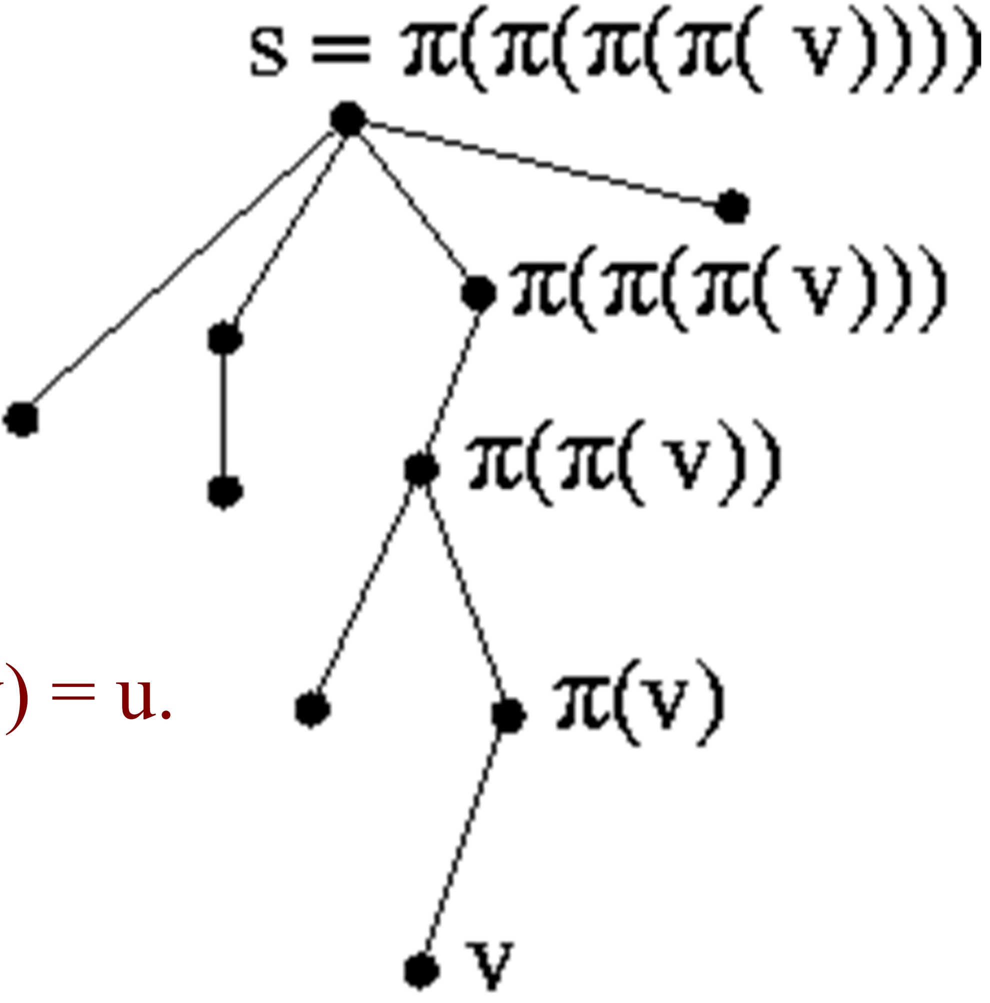


- The optimal substructure property
  - is a hallmark of both greedy and dynamic programming algorithms.
  - allows us to compute both shortest path distance and the shortest paths themselves by storing only one *d* value and one predecessor value per vertex.

Recovering the Shortest Path

For each node v, store predecessor of v in □(v).

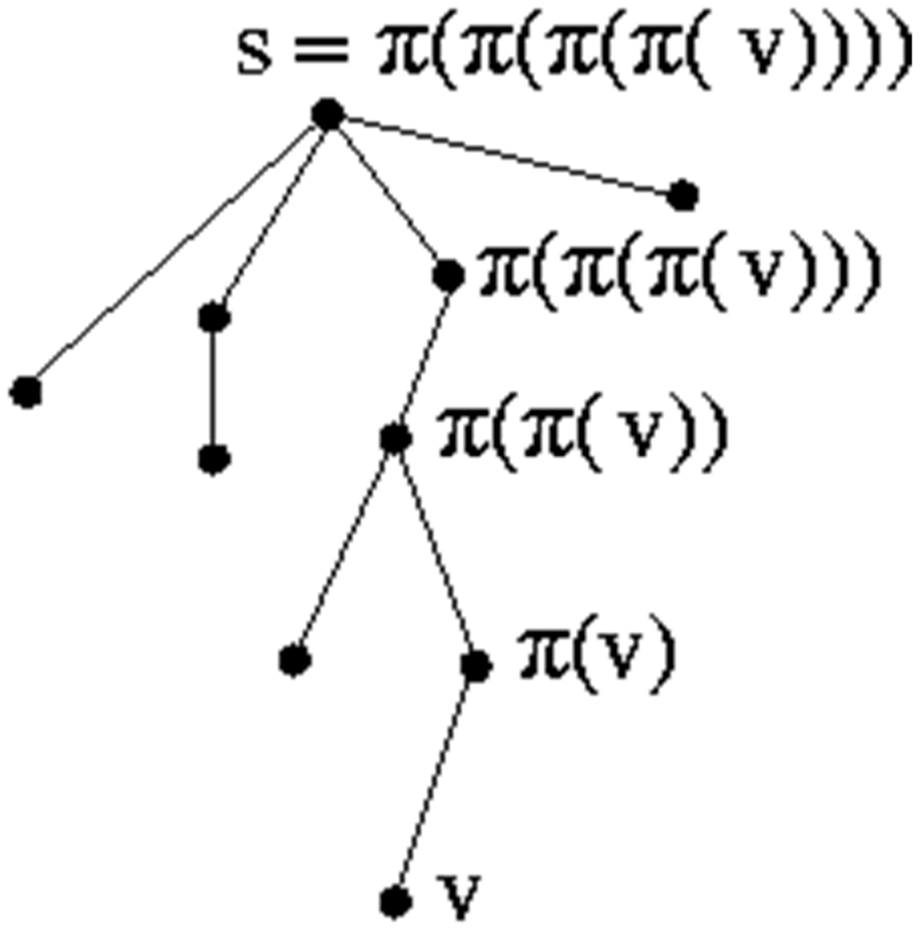




Predecessor of v is □(v) = u.

Recovering the Shortest Path

```
PRINT-PATH(G, s, v)
Precondition: s and v are vertices of graph G
Postcondition: the vertices on the shortest path from s to v have been printed in order
if v = s then
    print s
else if π[v] = NIL then
    print "no path from" s "to" v "exists"
else
    PRINT-PATH(G, s, π[v])
    print v
```



BFS Algorithm without Colours

BFS(G,s)

Precondition: *G* is a graph, *s* is a vertex in *G*

Postcondition: predecessors π[u] and shortest distance *d*[*u*] from *s* to each vertex *u* in *G* has been computed

```
    for each vertex u $\in V[G]$
        $d[u] \leftarrow \infty$
        $\pi[u] \leftarrow$ null
    $d[s] \leftarrow 0$
    Q.enqueue($s$)
    while Q $\neq \emptyset$
        u $\leftarrow$ Q.dequeue()
        for each $v \in$ Adj[$u$] //explore edge ($u$,$v$)
            if d[$v$] = $\infty$
                $d[v] \leftarrow d[u] + 1$
                $\pi[v] \leftarrow u$
                Q.enqueue($v$)
```

Outline

- BFS Algorithm
- BFS Application: Shortest Path on an unweighted graph
- Unweighted Shortest Path: Proof of Correctness